Coupling stochastic models of different time scales

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Abstract. A methodology is proposed for coupling stochastic models of hydrologic processes applying to different time scales so that time series generated by the different models be consistent. Given two multivariate time series, generated by two separate (unrelated) stochastic models of the same hydrologic process, each applying to a different time scale, a transformation is developed (referred to as a coupling transformation) that appropriately modifies the time series of the lower-level (finer) time scale so that this series be consistent with the time series of the higher-level (coarser) time scale without affecting the second-order stochastic structure of the former and also establishing appropriate correlations between the two time series. The coupling transformation is based on a developed generalized mathematical proposition, which ensures preservation of marginal and joint second-order statistics and of linear relationships between lower- and higher-level processes. Several specific forms of the coupling transformation are studied, from the simplest single variate to the full multivariate. In addition, techniques for evaluating parameters of the coupling transformation based on second order moments of the lower-level process are studied. Furthermore, two methods are proposed to enable preservation of the skewness of the processes, in addition to that of second-order statistics. The overall methodology can be applied to problems involving disaggregation of annual to seasonal and seasonal to subseasonal time scales, as well as problems involving finer time scales (e.g. daily – hourly), under the only requirement that a specific stochastic model is available for each involved time scale. The performance of the methodology is demonstrated by means of a detailed numerical example.

1 Introduction

Very often a hydrologic stochastic process must be studied in different time scales. Therefore, the problem arises of how to generate consistent time series both in a coarser, or higher-level, time scale and a finer, or lower-level, time scale. A trivial solution of this problem is to model the process in the lower-level time scale only, and then aggregate to derive the process in the higher-level time scale. However, there are reasons to avoid this solution and model the process in both time scales separately, each time focusing on different important statistical properties of the process [*Salas*, 1993, p. 19.32]. For instance, if the higher- and lower-level scales are annual and seasonal, respectively, the lower-level model may focus on the periodicity and short-term memory of the process whereas the higher-level model may focus on the long-term memory properties of the process. In other cases, the higher-level process may be the output of a specialized model (e.g., a meteorological rainfall prediction model) or known from measurements (e.g., daily rainfall measurements); apparently in such cases the aggregation approach cannot work, but rather disaggregation is needed.

Traditionally, this kind of problems is tackled by disaggregation models [*Valencia and Schaake*, 1972, 1973; *Mejia and Rousselle*, 1976; *Tao and Delleur*, 1976; *Hoshi and Burges*, 1979; *Lane*, 1979, 1982; *Salas et al.*, 1980; *Todini*, 1980; *Stedinger and Vogel*, 1984; *Pereira et al.*, 1984; *Stedinger et al.*, 1985; *Oliveira et al.*, 1988; *Grygier and Stedinger*, 1988, 1990; *Lane and Frevert*, 1990; *Santos and Salas*, 1992; *Koutsoyiannis*, 1992; *Salas*, 1993, p. 19.34; *Tarboton et al.*, 1998]. These are purposely-designed models to generate a process in the lower-level time scale given that in the higher-level. Specifically, they do not model the process of interest in the lower-level time scale itself, but rather they are hybrid schemes using simultaneously both time scales. Sometimes (owing to nonlinear transformations of variables) these models are not able to ensure consistency with the higher-level process. Then, adjusting procedures are necessary to restore consistency [*Grygier and Stedinger*, 1988, 1990; *Lane and Frevert*, 1990, p. V-22; *Koutsoyiannis and Manetas*, 1996].

However, there is the possibility of not designing and implementing a special model for disaggregation as a hybrid scheme incorporating both time scales. On the contrary, there may be available a model of the lower-level time scale with no reference to the higher-level time scale. The problem is then how a time series generated by the lower-level model can be modified so as to be consistent with a given higher-level time series, without affecting the stochastic structure implied by the lower-level model. (Practically, this is equivalent to the use of adjusting procedures mentioned before.) In a recent study, *Koutsoyiannis and Manetas* [1996] showed that this is possible without using any kind of disaggregation model but only using adjusting procedures on top of the separate lower-level model. Their adjusting procedures are accurate in the sense that they do not modify certain statistics of the lower-level process. In that study, a contemporaneous seasonal autoregressive (PAR(1)) model was used as the lower-level model.

The present study is a generalization of that by *Koutsoyiannis and Manetas* [1996] in several senses. Based on a generalized mathematical proposition, a wider transformation for modifying the lower-level time series, so as to be consistent with the higher-level time series, is introduced. Several forms of this transformation (referred to as coupling transformation) are studied. Apart from ensuring consistency with higher-level time series and reproducing second-order statistics of the lower-level variables within a certain period (higher-level time step), the transformation preserves lagged covariances of lower-level variables with lower- and higher-level variables of previous and next periods as well. Thus, a well-known defect of disaggregation models, i.e., their inconsistency in preserving accurately lagged covariances among lower- and higher-level variables [*Lane*, 1982; *Stedinger and Vogel*, 1984], is remedied. In addition, the most general form of the proposed coupling transformation is true multivariate, that is, it is applied simultaneously to all the variables of all locations involved in the problem examined, rather than adjusting the variables of each location separately. Furthermore, the methodology proposed can be applied not only to the simple PAR(1) model but to any type of stationary or seasonal stochastic model for any time scale, under the only requirement that a specific stochastic model is available for each involved time scale.

The theoretical background of the methodology proposed is presented in section 2. The specific forms of the coupling transformation are studied in section 3 while the methods for evaluating their parameters are given in section 4. The problem of preservation of the coefficients of skewness of the variables involved is examined separately in section 5. A numerical example that demonstrates the performance of the methodology is given in section 6 and conclusions are drawn in section 7. To increase readability, several mathematical derivations are excluded from the paper (given separately in an Appendix, available on microfiche).

2 Theoretical background

Let a hydrologic process, such as rainfall, runoff, etc., defined at n locations and studied in discrete time using two different time scales, the higher-level time scale with time step δ_{H} and the lower-level time scale with time step δ_{L} such that $k := \delta_{\mathrm{H}} / \delta_{\mathrm{L}}$ be an integer. We denote the higher- and lower-level discrete time processes by $\mathbf{Z}_p = [\overset{1}{Z}_p, \ldots, \overset{n}{Z}_p]^T$ and $\mathbf{X}_s = [\overset{1}{X}_s, \ldots, \overset{n}{X}_s]^T$, respectively, where superscript T denotes the transpose of a vector or matrix and subscripts p and s are integer time indices that stand for period and subperiod, respectively, with common origin (i.e., at the time origin $p = 0$ and $s = 0$). Generally, in this paper we use upper case letters for random variables, and lower case letters for values, parameters, or constants. Furthermore, we use bold letters for arrays or vectors, and normal letters for their elements. Higher- and lower-level processes are related by

$$\sum_{s = (p-1)k+1}^{pk} \mathbf{X}_s = \mathbf{Z}_p \tag{1}$$

We assume that two separate stochastic models have been built, one for the higher-level process $\mathbf{Z}_p$ and one for the lower-level process $\mathbf{X}_s$, without link or reference between them. To increase readability, we can refer to the simple example where $\mathbf{Z}_p$ and $\mathbf{X}_s$ represent the annual and monthly flows at n locations, modeled as an AR(1) (autoregressive process of order 1) and a PAR(1) (periodic or seasonal autoregressive process of order 1), respectively. These models are expressed by

$$\mathbf{Z}_p = \mathbf{a}' \, \mathbf{Z}_{p-1} + \mathbf{b}' \, \mathbf{V}'_p \tag{2}$$

and

$$\mathbf{X}_s = \mathbf{a}_s \, \mathbf{X}_{s-1} + \mathbf{b}_s \, \mathbf{V}_s \tag{3}$$

where all $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{a}_s$, and $\mathbf{b}_s$ are $(n \times n)$ matrices of parameters and $\mathbf{V}'_p$ and $\mathbf{V}_s$ ($s, p = \ldots, 0, 1, 2, \ldots$) are vectors of innovations (independent, both in time and location, random variables) with size n. The time indices s, p can take any integer value but in our example the parameters $\mathbf{a}_s$ and $\mathbf{b}_s$ are periodic functions of s with period k whereas $\mathbf{a}'$ and $\mathbf{b}'$ do not vary with p.

We emphasize that models (2) and (3) and the relevant assumptions are given here just as simple model examples of a generalized methodology that can be combined with any type of multivariate stochastic models with any distribution functions and perform in any time scale. In fact, as no assumption is implied about the involved models, the methodology can be combined with linear and nonlinear stochastic models also including generalized autocovariance models [*Koutsoyiannis*, 2000], nonparametric models [*Lall and Sharma*, 1996; *Sharma et al.*,1997] and hybrid models [*Srinivas and Srinivasan*, 2000]. Moreover, the modeling time scales need not be annual and monthly as in the examples used herein, but can be much finer such as daily and hourly, even though distribution functions at those time scales are much more asymmetric.

We also notice that higher- and lower-level models need not be fully compatible. For example, models (2) and (3) are not fully compatible, as the covariance structure implied by (3) for the higher-level process $\mathbf{Z}_p$ (determined by invoking (1)) is not identical with that implied by (2) (it can be easily verified that the sum of AR(1) or PAR(1) processes cannot be an AR(1) process). If the models were fully compatible, the problem examined would be trivial, as the lower-level model would actually incorporate the higher-level model. Thus, the problem acquires its interest in case of (partially) incompatible models each focusing on different important statistical properties of the stochastic processes examined. For instance, the lower-level model may focus on the periodicity and the short-term memory of the process whereas the higher-level model may focus on the long-term memory properties of the process. Apparently, (2) is not adequate for the latter case and more complex models such as that proposed by *Koutsoyiannis* [2000] must be used instead.

Let us assume that a time series $\mathbf{z}_p$ of the process $\mathbf{Z}_p$ has been generated using model (2) (or any other, linear or nonlinear, parametric or nonparametric, model, or even, it has been acquired from measurements) and another time series $\tilde{\mathbf{x}}_s$ of the process $\mathbf{X}_s$ has been generated using (3) (or another appropriate stochastic model). The latter time series has been generated independently of the former and, therefore, $\tilde{\mathbf{x}}_s$ do not add up to $\mathbf{z}_p$, as demanded by the additive property (1), but to some other quantities, which we will denote $\tilde{\mathbf{z}}_p$. We wish to modify the series $\tilde{\mathbf{x}}_s$ thus producing a series $\mathbf{x}_s$ consistent with $\mathbf{z}_p$, in the sense that $\mathbf{x}_s$ and $\mathbf{z}_p$ obey (1), without affecting the stochastic structure of the lower-level time series. For convenience we will assume that $\tilde{\mathbf{x}}_s$ is a realization of a stochastic process $\tilde{\mathbf{X}}_s$, identical to $\mathbf{X}_s$ (e.g., following (3)) and the series $\tilde{\mathbf{z}}_p$ is a realization of a process $\tilde{\mathbf{Z}}_p$ defined as the sum of $\tilde{\mathbf{X}}_s$. In the ideal case that the processes $\mathbf{X}_s$ and $\mathbf{Z}_p$ are fully compatible, $\tilde{\mathbf{Z}}_p$ will be identical to $\mathbf{Z}_p$, but, as discussed above, this is not the case in general (note that $\tilde{\mathbf{Z}}_p$ is derived as a summation of the lower-level process whereas $\mathbf{Z}_p$ corresponds to the higher-level model regardless of the lower-level model). We seek for a transformation $\mathbf{f}(\tilde{\mathbf{X}}_s, \tilde{\mathbf{Z}}_p, \mathbf{Z}_p)$ whose outcome is a process identical to $\mathbf{X}_s$ and consistent to $\mathbf{Z}_p$ (it satisfies (1)). We will use the symbol $\mathbf{X}_s$ for the outcome of this transformation (i.e., $\mathbf{X}_s = \mathbf{f}(\tilde{\mathbf{X}}_s, \tilde{\mathbf{Z}}_p, \mathbf{Z}_p)$) and we will call this transformation a coupling transformation. With the followed notation we have two couples of processes, the auxiliary processes ($\tilde{\mathbf{X}}_s$, $\tilde{\mathbf{Z}}_p$) and the "actual" processes ($\mathbf{X}_s$, $\mathbf{Z}_p$); in each couple, lower- and higher-level processes are consistent (i.e., they satisfy (1)), but members of different couples are inconsistent. A schematic representation of the four processes involved, their links, and the steps followed to construct the "actual" lower-level process $\mathbf{X}_s$, consistent with $\mathbf{Z}_p$, is shown in Figure 1.

We can determine an appropriate linear form of the coupling transformation based on the following general proposition, specific forms of which we will extract and utilize in next sections. For the generalized presentation of the Proposition given below, the reader may have in mind that the vectors $\tilde{\mathbf{X}}$ and $\mathbf{X}$ contain numerous items of the auxiliary and actual lower-level processes, respectively, and the vectors $\tilde{\mathbf{Y}}$ and $\mathbf{Y}$ contain items of the higher-level processes and other variables that will be specified later. The additive property (1) is represented here by a more generalized linear relationship of the form of equation (6) below.

Proposition: Let $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ be vectors of random variables with means $E[\tilde{\mathbf{X}}]$ and $E[\tilde{\mathbf{Y}}]$, variance-covariance matrices Cov[$\tilde{\mathbf{X}}$, $\tilde{\mathbf{X}}$] and Cov[$\tilde{\mathbf{Y}}$, $\tilde{\mathbf{Y}}$], respectively, and joint covariance matrix $\mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]$. Let also $\mathbf{Y}$ be a vector of stochastic variables independent of $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ with means and variance-covariance matrix identical to that of $\tilde{\mathbf{Y}}$. Define

$$\mathbf{X} := \tilde{\mathbf{X}} + \mathbf{h}\,(\mathbf{Y} - \tilde{\mathbf{Y}}) \tag{4}$$

where $\mathbf{h}$ is a matrix of parameters given by

$$\mathbf{h} := \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]\ \{\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\}^{-1} \tag{5}$$

Then:

(a) $\mathbf{X}$ has mean and variance-covariance matrix identical to those of $\tilde{\mathbf{X}}$, and joint covariance matrix with $\mathbf{Y}$ identical to that of $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$.

(b) Any linear relationships that hold among $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$, which can be written in the form

$$\mathbf{g}_{\mathbf{X}}^{T}\,\tilde{\mathbf{X}} = \mathbf{g}_{\mathbf{Y}}^{T}\,\tilde{\mathbf{Y}} \tag{6}$$

where $\mathbf{g}_{\mathbf{X}}$ and $\mathbf{g}_{\mathbf{Y}}$ are matrices (or vectors, in case of a single linear relationship) of coefficients, hold also among $\mathbf{X}$ and $\mathbf{Y}$, that is

$$\mathbf{g}_{\mathbf{X}}^{T}\,\mathbf{X} = \mathbf{g}_{\mathbf{Y}}^{T}\,\mathbf{Y} \tag{7}$$

(c) The conditional variance of any element X_i of the vector $\mathbf{X}$, given $\mathbf{Y} = \mathbf{y}$, is

$$\mathrm{Var}[X_i \mid \mathbf{Y} = \mathbf{y}] = \mathrm{Var}[\tilde{X}_i] - \mathrm{Cov}[\tilde{X}_i, \tilde{\mathbf{Y}}]\ \{\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\}^{-1}\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{X}_i] \tag{8}$$

and is identical to the least mean square prediction error of X_i from $\mathbf{Y}$.

The proof of the Proposition is given in Appendix A1. We mention here an interesting intermediate result regarding the proof of item (b) of the Proposition: if (6) holds, $\mathbf{g}_{\mathbf{X}}$ and $\mathbf{g}_{\mathbf{Y}}$ affect the covariance matrices $\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]$ and $\mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]$ and consequently $\mathbf{h}$, so that finally

$$\mathbf{g}_{\mathbf{Y}}^{T} = \mathbf{g}_{\mathbf{X}}^{T}\,\mathbf{h} \tag{9}$$

We also notice that, given the equality of covariances between the couples ($\tilde{\mathbf{X}}$, $\tilde{\mathbf{Y}}$) and ($\mathbf{X}$, $\mathbf{Y}$) we can substitute any covariance matrix of the first couple with the corresponding of the second couple; for example, we can write (5) as $\mathbf{h} := \mathrm{Cov}[\mathbf{X}, \mathbf{Y}]\ \{\mathrm{Cov}[\mathbf{Y}, \mathbf{Y}]\}^{-1}$.

Equations (4) and (5) offer the basis to develop the coupling transformation, as we will see in the next section. Item (c) of the Proposition, although not used directly in developing the coupling transformation, ensures that (4) provides the best possible conditional estimate of $\mathbf{X}$ given $\mathbf{Y}$ in a least-squares sense.

3 Specific forms of coupling transformation

In this section we will develop several forms of the coupling transformation, starting from the simplest case of a single variate model and proceeding towards more complex cases. For mathematical convenience, the transformation will extend to the lower-level variables of one period only (rather than extending to all simulated periods simultaneously) yet considering the necessary links to previous and next periods. For notational convenience, we will assume that the time origin coincides with the origin of the examined period so that $p = 1$. Thus, we will write (1) as

$$\sum_{s=1}^{k} \mathbf{X}_s = \mathbf{Z}_1 \tag{10}$$

We introduce the following notational conventions of covariances among lower- and/or higher-level variables:

$$\begin{aligned} &\boldsymbol{\sigma}_{sr} := \mathrm{Cov}[\mathbf{X}_s, \mathbf{X}_r] \equiv \boldsymbol{\sigma}_{rs}^T, \qquad &&\boldsymbol{\varphi}_{pr} := \mathrm{Cov}[\mathbf{Z}_p, \mathbf{Z}_r] \equiv \boldsymbol{\varphi}_{rp}^T, \\ &\boldsymbol{\tau}_{sp} := \mathrm{Cov}[\mathbf{X}_s, \mathbf{Z}_p], \qquad &&\boldsymbol{\tau}'_{ps} := \mathrm{Cov}[\mathbf{Z}_p, \mathbf{X}_s] \equiv \boldsymbol{\tau}_{sp}^T \end{aligned} \tag{11}$$

The evaluation of these parameters will be discussed in section 4.

3.1 Preserving the additive property

In the simplest case, we assume a single site model with lower-level variables $X_1, \ldots, X_k$ adding up to the higher-level variable Z_1. We apply the Proposition of section 2 setting $\mathbf{X} = [X_1, \ldots, X_k]^T$ and $\mathbf{Y} = [Z_1]$. In the single site case examined we have $\sigma_{sr} = \mathrm{Cov}[X_s, X_r] \equiv \sigma_{rs}$, $\tau_{sp} = \mathrm{Cov}[X_s, Z_p] \equiv \tau'_{ps}$, $\varphi_{pr} = \mathrm{Cov}[Z_p, Z_r] \equiv \varphi_{rp}$. The additive property (10) can be written in the form (6) with $\mathbf{g}_\mathbf{X}^T = [1, 1, ..., 1]$ and $\mathbf{g}_\mathbf{Y}^T = 1$. The parameter matrix $\mathbf{h}$ is (from (4))

$$\mathbf{h} = \frac{1}{\varphi_{11}} [\tau_{11}, \ldots, \tau_{k1}]^T \tag{12}$$

and thus, the coupling transformation (4) can be written for each subperiod s as

$$X_s := \tilde{X}_s + \frac{\tau_{s1}}{\varphi_{11}} (Z_1 - \tilde{Z}_1) \tag{13}$$

This is the simple adjusting procedure developed by *Koutsoyiannis and Manetas* [1996]. Note that each of the coefficients τ_{s1} / φ_{11} for a specific s represents the ratio of the covariance of each lower-level variable X_s with the higher-level variable Z_1 (τ_{s1}) to the variance of the higher-level variable Z_1 (φ_{11}). Thus, (13) distributes the departure $(Z_1 - \tilde{Z}_1)$ of the additive property to each lower-level variable, proportionally to the covariance of this lower-level variable with the higher-level variable. Note also that the covariances τ_{s1} for all s add up to the variance φ_{11} (see also section 4) and thus the coefficients τ_{s1} / φ_{11} for all s add up to 1, as they should. Thus, the sum of all X_s will equal Z_1 regardless of the values of $\tilde{X}_s$, i.e. the preservation of the additive property is ever assured. The special case where the lower-level variables are independent (the process X_s is white noise), although unusual, provides better understanding of the rationale of (13). In this case $\tau_{s\,1}$ equals the variance of the lower-level variable X_s so that the distribution of the departure $(Z_1 - \tilde{Z}_1)$ to each lower-level variable becomes proportional to the variance of the variable. Interestingly, *Grygier and Stedinger* [1988] and *Lane and Frevert* [1990, p. V-22] had proposed a similar empirical adjusting procedure but using the standard deviation in place of the variance of each of the lower-level variables. We must emphasize, however, that the exact transformation that assures preservation of the additive property, means and second order moments of the process in the general case of dependent variables is expressed as in (13) in terms of the covariances $\tau_{s\,1}$ rather than variances or standard deviations of the different lower-level variables.

3.2 Linking with lower-level variables of the previous period

The above simple transformation preserves the additive property and correlations of lower-level variables within the examined period. However, it does not preserve explicitly the correlations of lower-level variables with subperiods of previous periods. For example, it does not preserve explicitly the correlation of the first lower-level variable X_1 with the last lower-level variable of the previous period X_0. This can be easily remedied by setting $\mathbf{X} = [X_1, \ldots, X_k]^T$ and $\mathbf{Y} = [X_0, Z_1]^T$. In this case, $\mathbf{g}_\mathbf{X}^T = [1, 1, ..., 1]$ and $\mathbf{g}_\mathbf{Y}^T = [0, 1]$ so that $\mathbf{g}_\mathbf{X}^T \mathbf{X}$ be the sum of all lower-level variables and $\mathbf{g}_\mathbf{Y}^T \mathbf{Y}$ be the higher-level variable Z_1. The parameter matrix $\mathbf{h}$ is

$$\mathbf{h} = \begin{bmatrix} \sigma_{10} & \tau_{11} \\ \vdots & \vdots \\ \sigma_{k\,0} & \tau_{k\,1} \end{bmatrix} \begin{bmatrix} \sigma_{00} & \tau_{01} \\ \tau_{01} & \varphi_{11} \end{bmatrix}^{-1} \tag{14}$$

Thus, the coupling transformation (4) can be written for each subperiod s as

$$X_s := \tilde{X}_s + \frac{1}{\sigma_{00}\,\varphi_{11} - \tau_{01}^2} [(\varphi_{11}\,\sigma_{s\,0} - \tau_{01}\,\tau_{s1})\,(X_0 - \tilde{X}_0) + (\sigma_{00}\,\tau_{s1} - \tau_{01}\,\sigma_{s\,0})\,(Z_1 - \tilde{Z}_1)] \tag{15}$$

This notion can be extended to include a greater number of previous lower-level variables, as we will see in subsection 3.3. Here an explanation of the rationale of the different terms of (15) is no longer simple as it was in (13) (this in even more the case for the equations of the following subsections 3.3 and 3.4). We can only say that (15) performs two kinds of adjustment to the auxiliary lower-level variables $\tilde{X}_s$: First, it distributes the departure $(Z_1 - \tilde{Z}_1)$ among the different lower-level variables so as to restore the additive property. Second, it modifies $\tilde{X}_s$ in proportion to the departure $(X_0 - \tilde{X}_0)$ so as to restore the dependence with lower-level variables of the previous period. It may be easily verified that the coefficients used to distribute $(Z_1 - \tilde{Z}_1)$ add up to 1, as they should, whereas the coefficients used for $(X_0 - \tilde{X}_0)$ add up to 0, as they should, too.

3.3 Linking with next higher-level variables

Linking the lower-level variables of the current period with those of the previous period, in the way discussed in subsection 3.2, may be regarded as linking with lower-level variables of the next subperiod, as well. Specifically, at the stage of generating the lower-level variables of the next period the correlation with the lower-level variables of the current period will be preserved in the manner discussed in previous subsection.

However, this is not absolutely correct, because in this manner the lagged correlations between lower- and higher-level variables are not considered explicitly. As shown by *Stedinger and Vogel* [1984] the departures in preserving such lagged correlations are responsible for inconsistencies in preserving correlations between lower-level variables of different periods; this problem was first reported by *Lane* [1982], and contributions to overcome it were made by *Stedinger and Vogel* [1984], *Lin* [1990], *Koutsoyiannis* [1992] and *Koutsoyiannis and Manetas* [1996].

The developed general proposition allows for an effective tackling of this problem. In addition to correlations with the previous lower-level variables, discussed in the previous subsection, we will also consider the preservation of correlations between the lower-level variables of the current period and the higher-level variable of the next period. We note that the correlation of the former with the higher-level variables of the previous periods has been already considered indirectly (through correlations with the corresponding lower-level variables) whereas the correlation with the higher-level variable of the current period has been incorporated explicitly (through the coupling transformation).

We will distinguish between two cases regarding the succession of generation steps of higher- and lower-level variables. In the first case, all higher-level variables of all periods are generated before the generation of lower-level variables. In the second case, the generation of lower-level variables of one period follows the generation of the higher-level variable of that period and precedes that of the next period.

In the first case, at the step of generating the lower-level variables of the current period, the higher-level variable of the next period (Z_2) is already known and the correlation with it must be consider and preserved. This may be essential especially when this correlation is high (e.g., for fine time scales). To this aim we must append Z_2 to the vector $\mathbf{Y}$ again setting $\mathbf{X} = [X_1, \ldots, X_k]^T$. In addition, to acquire a more generalized solution than that of subsection 3.2, which was appropriate for the specific model (3), we append to $\mathbf{Y}$ a number q (depending on the lower-level model used) of lower-level variables prior to X_0 so that finally $\mathbf{Y} = [X_{-q}, \ldots, X_0, Z_1, Z_2]^T$. The vectors $\mathbf{g_X}$ and $\mathbf{g_Y}$ become $\mathbf{g}_\mathbf{X}^T = [1, 1, ..., 1]$ and $\mathbf{g}_\mathbf{Y}^T = [0, \ldots, 0, 1, 0]$. The parameter matrix $\mathbf{h}$ is

$$\mathbf{h} = \begin{bmatrix} \sigma_{1,-q} & \cdots & \sigma_{10} & \tau_{11} & \tau_{12} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \sigma_{k,-q} & \cdots & \sigma_{k,0} & \tau_{k,1} & \tau_{k,2} \end{bmatrix} \begin{bmatrix} \sigma_{-q,-q} & \cdots & \sigma_{-q,0} & \tau_{-q,1} & \tau_{-q,2} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \sigma_{0,-q} & \cdots & \sigma_{00} & \tau_{01} & \tau_{02} \\ \tau'_{1,-q} & \cdots & \tau'_{10} & \varphi_{11} & \varphi_{12} \\ \tau'_{2,-q} & \cdots & \tau'_{20} & \varphi_{21} & \varphi_{22} \end{bmatrix}^{-1} \tag{16}$$

The coupling transformation (4) can be written for each subperiod s as

$$X_s := \widetilde{X}_s + \mathbf{h}_s\,[(X_{-q} - \widetilde{X}_{-q}), \ldots, (X_0 - \widetilde{X}_0), (Z_1 - \widetilde{Z}_1), (Z_2 - \widetilde{Z}_2)]^T \tag{17}$$

where $\mathbf{h}_s$ is the sth row of $\mathbf{h}$.

In the second case mentioned above (which is met rather rarely), at the step of generation of the lower-level variables of the current period, the higher-level variable of the next period Z_2 is not known and thus, the analysis of subsection 3.2 suffices. Just before that step, the higher-level variable Z_1 of the current period is to be generated. At this time, the lower-level variables of the previous periods (X_0, X_{-1}, …) are already known and correlation with them must be preserved. However, this is not done automatically by the higher-level model itself (e.g., by (2)). Using the general Proposition we can remedy this problem as well, if we set $\mathbf{X} = [Z_1]$ and $\mathbf{Y} = [X_{-q}, \ldots, X_0]^T$, i.e., the vector of $q + 1$ lower-level variables of the previous periods. In this case, $q + 1$ can be chosen equal to k, the number of lower-level variables of one period, but it can be lower or greater than this value as well, depending on how large the correlation of higher- to lagged lower-level variables is. The parameter matrix now is

$$\mathbf{h_Z} = [\tau'_{1,-q} \ldots \tau'_{10}] \begin{bmatrix} \sigma_{-q,-q} & \cdots & \sigma_{-q,0} \\ \vdots & \ddots & \vdots \\ \sigma_{0,-q} & \cdots & \sigma_{00} \end{bmatrix}^{-1} \tag{18}$$

and the coupling transformation is

$$Z_1 := \widetilde{Z}_1 + \mathbf{h_Z}\,[(X_{-q} - \widetilde{X}_{-q}), \ldots, (X_0 - \widetilde{X}_0)]^T \tag{19}$$

The generation of the lower-level variables of the current period follows that of the higher-level variable. This is done by (14) and (15), if only one previous lower-level variable is considered, or otherwise by the more general relationship

$$X_s := \widetilde{X}_s + \mathbf{h}_s\,[(X_{-q} - \widetilde{X}_{-q}), \ldots, (X_0 - \widetilde{X}_0), (Z_1 - \widetilde{Z}_1)]^T \tag{20}$$

where $\mathbf{h}_s$ is the sth row of the matrix $\mathbf{h}$ that is now given by

$$\mathbf{h} = \begin{bmatrix} \sigma_{1,-q} & \cdots & \sigma_{10} & \tau_{11} \\ \vdots & \ddots & \vdots & \vdots \\ \sigma_{k,-q} & \cdots & \sigma_{k,0} & \tau_{k,1} \end{bmatrix} \begin{bmatrix} \sigma_{-q,-q} & \cdots & \sigma_{-q,0} & \tau_{-q,1} \\ \vdots & \ddots & \vdots & \vdots \\ \sigma_{0,-q} & \cdots & \sigma_{00} & \tau_{01} \\ \tau'_{1,-q} & \cdots & \tau'_{10} & \varphi_{11} \end{bmatrix}^{-1} \tag{21}$$

Here, (20) and (21) have been derived from (17) and (16), respectively, by omitting all elements referring to Z_2.

3.4 Multivariate case

The above forms of the coupling transformation can be applied location by location in the case of a multivariate process. However, in this manner, the cross-correlations of the lower-level variables will be altered by the single-location coupling transformations of different locations. The same coupling transformations can be formulated in a true multivariate form, so that cross-correlations be explicitly preserved. This is a very simple task, as it suffices to write the same relationships in multivariate form. We will give here the multivariate version of the most general case of subsection 3.3; the other cases are remedied in a similar manner.

The vector **X** is formed by appending all vectors of lower-level variables of the current period, and the vector **Y** is constructed in a similar manner, i.e.,

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_k \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} \mathbf{X}_{-q} \\ \vdots \\ \mathbf{X}_0 \\ \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} \tag{22}$$

Thus, **X** and **Y** have $k\,n$ and $(q + 3)\,n$ elements, respectively. The matrices $\mathbf{g}_\mathbf{X}$ and $\mathbf{g}_\mathbf{Y}$, needed to express the additive property in the multivariate form (10), are constructed as in subsection 3.3 but replacing 1 with the $n \times n$ identity matrix **I** and 0 with the $n \times n$ zero matrix **O**, that is

$$\mathbf{g}_\mathbf{X}^T = [\mathbf{I}, \mathbf{I}, ..., \mathbf{I}] \text{ and } \mathbf{g}_\mathbf{Y}^T = [\mathbf{O}, \ldots, \mathbf{O}, \mathbf{I}, \mathbf{O}] \tag{23}$$

For example, in a problem with $k = 3$ lower-level variables, $n = 2$ locations, and $q = 0$, the relevant vectors and matrices become

$$\mathbf{X} = \begin{bmatrix} X_1^1 \\ X_1^2 \\ X_2^1 \\ X_2^2 \\ X_3^1 \\ X_3^2 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} X_0^1 \\ X_0^2 \\ Z_1^1 \\ Z_1^2 \\ Z_2^1 \\ Z_2^2 \end{bmatrix}, \quad \mathbf{g}_\mathbf{X}^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{g}_\mathbf{Y}^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{24}$$

It can be directly verified from this example that the relationship (7) (i.e., $\mathbf{g}_\mathbf{X}^T\,\mathbf{X} = \mathbf{g}_\mathbf{Y}^T\,\mathbf{Y}$) is identical to the additive property (10).

The parameter matrix **h** is constructed as in subsection 3.3 but replacing each scalar covariance σ, τ, τ' or φ with its corresponding $n \times n$ matrix $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, $\boldsymbol{\tau}'$ or $\boldsymbol{\varphi}$, respectively. Thus

$$\mathbf{h} = \begin{bmatrix} \boldsymbol{\sigma}_{1,-q} & \cdots & \boldsymbol{\sigma}_{10} & \boldsymbol{\tau}_{11} & \boldsymbol{\tau}_{12} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \boldsymbol{\sigma}_{k,-q} & \cdots & \boldsymbol{\sigma}_{k,0} & \boldsymbol{\tau}_{k,1} & \boldsymbol{\tau}_{k,2} \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_{-q,-q} & \cdots & \boldsymbol{\sigma}_{-q,0} & \boldsymbol{\tau}_{-q,1} & \boldsymbol{\tau}_{-q,2} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \boldsymbol{\sigma}_{0,-q} & \cdots & \boldsymbol{\sigma}_{00} & \boldsymbol{\tau}_{01} & \boldsymbol{\tau}_{02} \\ \boldsymbol{\tau}'_{1,-q} & \cdots & \boldsymbol{\tau}'_{10} & \boldsymbol{\varphi}_{11} & \boldsymbol{\varphi}_{12} \\ \boldsymbol{\tau}'_{2,-q} & \cdots & \boldsymbol{\tau}'_{20} & \boldsymbol{\varphi}_{21} & \boldsymbol{\varphi}_{22} \end{bmatrix}^{-1} \tag{25}$$

The coupling transformation (4) is

$$\mathbf{X} := \tilde{\mathbf{X}} + \mathbf{h}\,[(\mathbf{X}_{-q}^T - \tilde{\mathbf{X}}_{-q}^T), \ldots, (\mathbf{X}_0^T - \tilde{\mathbf{X}}_0^T), (\mathbf{Z}_1^T - \tilde{\mathbf{Z}}_1^T), (\mathbf{Z}_2^T - \tilde{\mathbf{Z}}_2^T)]^T \tag{26}$$

In the second case mentioned in subsection 3.3 we have again two steps. At the first step, concerning the generation of the higher-level variables, the corresponding multivariate variables are $\mathbf{X} = \mathbf{Z}_1$ and $\mathbf{Y} = [\mathbf{X}_{-q}^T, \ldots, \mathbf{X}_0^T]^T$. The parameter matrix $\mathbf{h}_\mathbf{Z}$ now is

$$\mathbf{h}_\mathbf{Z} = [\boldsymbol{\tau}'_{1,-q} \ldots \boldsymbol{\tau}'_{10}] \begin{bmatrix} \boldsymbol{\sigma}_{-q,-q} & \cdots & \boldsymbol{\sigma}_{-q,0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\sigma}_{0,-q} & \cdots & \boldsymbol{\sigma}_{00} \end{bmatrix}^{-1} \tag{27}$$

and the coupling transformation is

$$\mathbf{Z}_1 := \tilde{\mathbf{Z}}_1 + \mathbf{h}_\mathbf{Z}\,[(\mathbf{X}_{-q}^T - \tilde{\mathbf{X}}_{-q}^T), \ldots, (\mathbf{X}_0^T - \tilde{\mathbf{X}}_0^T)]^T \tag{28}$$

At the second step, concerning the generation of the lower-level variables, the vectors of variables are $\mathbf{X} = [\mathbf{X}_1^T, \ldots, \mathbf{X}_k^T]^T$ and $\mathbf{Y} = [\mathbf{X}_{-q}^T, \ldots, \mathbf{X}_0^T, \mathbf{Z}_1^T]^T$, the parameter matrix is

$$\mathbf{h} = \begin{bmatrix} \boldsymbol{\sigma}_{1,-q} & \cdots & \boldsymbol{\sigma}_{10} & \boldsymbol{\tau}_{11} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{\sigma}_{k,-q} & \cdots & \boldsymbol{\sigma}_{k,0} & \boldsymbol{\tau}_{k,1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_{-q,-q} & \cdots & \boldsymbol{\sigma}_{-q,0} & \boldsymbol{\tau}_{-q,1} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{\sigma}_{0,-q} & \cdots & \boldsymbol{\sigma}_{00} & \boldsymbol{\tau}_{01} \\ \boldsymbol{\tau}'_{1,-q} & \cdots & \boldsymbol{\tau}'_{10} & \boldsymbol{\varphi}_{11} \end{bmatrix}^{-1} \tag{29}$$

and the coupling transformation is

$$\mathbf{X} := \tilde{\mathbf{X}} + \mathbf{h}\,[(\mathbf{X}_{-q}^T - \tilde{\mathbf{X}}_{-q}^T), \ldots, (\mathbf{X}_0^T - \tilde{\mathbf{X}}_0^T), (\mathbf{Z}_1^T - \tilde{\mathbf{Z}}_1^T)]^T \tag{30}$$

4 Evaluation of parameters of coupling transformation

We have seen in section 3 that all forms of the coupling transformation involve three categories of parameters, defined in (11). Namely, these are: (a) covariances between lower-level variables, denoted by σ; (b) covariances between higher-level variables, denoted by φ; and (c) covariances between lower- and higher-level variables, denoted by τ or τ'.

A first option to evaluate these parameters would be to refer to the historical data. This, however, must be avoided for several reasons, i.e., (a) because in this way we would introduce a vast number of parameters depended on, and estimated from, the data, (b) because usually historical data records are limited and inadequate to estimate such a large parameter set; (c) because such a large parameter set, if estimated from historical data, may not be consistent with the higher- or lower-level models, which are usually expressed in terms of a parameter set as parsimonious as possible. The alternative is to let models determine the parameters (more specifically, the lower-level model, as it will be explained later). There are two options to do this, one numerical and one analytical.

The numerical option is based on stochastic simulation and is fully generalized, as it can perform with any type of lower-level model: We can generate a synthetic data record of lower-level variables $\tilde{\mathbf{X}}$ with an appropriate length and aggregate it to obtain the higher-level variables $\tilde{\mathbf{Z}}$. As covariances between $\tilde{\mathbf{X}}$ and/or $\tilde{\mathbf{Z}}$ equal those of $\mathbf{X}$ and/or $\mathbf{Z}$, we can use these synthetic data records to estimate directly the parameters. This option has the advantage of being simple and independent of the type of the model. However, it has the disadvantages of the approximate character of estimations and the computational effort needed.

The analytical option is case-specific and uses the properties of the lower-level model chosen to determine the needed parameters theoretically. Owing to its exact character and the fast evaluation of parameters, this option is the most preferable whenever analytical equations can be established for the model chosen. Below, we will give the equations that are necessary to evaluate the needed parameters for a list of very common lower-level models of the literature [e.g., *Salas et al.*, 1980; *Bras and Rodriguez-Iturbe*, 1985; *Lane and Frevert*, 1990; *Grygier and Stedinger*, 1990; *Salas*, 1993]. The derivations of equations are given in the Appendix A3 and may serve as a basis for extending the list given here with more models.

For the simple PAR(1) example defined by (3), the covariance of lower-level variables for any lag $(s - r)$ is given by

$$\boldsymbol{\sigma}_{sr} := \mathrm{Cov}[\mathbf{X}_s, \mathbf{X}_r] = \mathbf{a}_s\, \mathbf{a}_{s-1} \cdots \mathbf{a}_{r+1}\, \boldsymbol{\sigma}_{rr}\,, \quad s > r \tag{31}$$

so that all lagged covariances among lower-level variables $\boldsymbol{\sigma}_{sr}$ are determined in terms of lag-zero cross-covariances $\boldsymbol{\sigma}_{ss}$ and the model parameters $\mathbf{a}_s$. Similar (although somehow more complex) is the situation with other common hydrologic stochastic models. Thus, the PAR(2) model, expressed by

$$\mathbf{X}_s = \mathbf{a}_s\, \mathbf{X}_{s-1} + \mathbf{e}_s\, \mathbf{X}_{s-2} + \mathbf{b}_s\, \mathbf{V}_s \tag{32}$$

where all $\mathbf{a}_s$, $\mathbf{e}_s$ and $\mathbf{b}_s$ are $(n \times n)$ matrices of parameters, results in

$$\boldsymbol{\sigma}_{sr} = \mathbf{a}_s\, \boldsymbol{\sigma}_{s-1,r} + \mathbf{e}_s\, \boldsymbol{\sigma}_{s-2,r}\,, \quad s > r \tag{33}$$

By applying this relationship recursively, for $s = r + 1$, $r + 2$, …, we can find any lagged covariance of lower-level variables in terms of lag-zero and lag-one covariance matrices ($\boldsymbol{\sigma}_{ss}$ and $\boldsymbol{\sigma}_{s-1,s}$) and the model parameters $\mathbf{a}_s$ and $\mathbf{e}_s$.

Similarly, the PARMA(1, 1) model, expressed by

$$\mathbf{X}_s = \mathbf{a}_s\, \mathbf{X}_{s-1} + \mathbf{b}_s\, \mathbf{V}_s + \mathbf{e}_s\, \mathbf{V}_{s-1} \tag{34}$$

where $\mathbf{a}_s$, $\mathbf{b}_s$ and $\mathbf{e}_s$ are $(n \times n)$ matrices of parameters, results in

$$\begin{aligned} \boldsymbol{\sigma}_{sr} &= \mathbf{a}_s\, \boldsymbol{\sigma}_{rr} + \mathbf{e}_s\, \mathbf{b}_r^T\,, \quad s = r + 1 \\ \boldsymbol{\sigma}_{sr} &= \mathbf{a}_s\, \boldsymbol{\sigma}_{s-1,r}\,, \quad s > r + 1 \end{aligned} \tag{35}$$

By applying this relationship recursively, for $s = r + 1$, $r + 2$, …, we can find any lagged covariance of lower-level variables in terms of the lag-zero covariance matrix $\boldsymbol{\sigma}_{ss}$ and the model parameters $\mathbf{a}_s$, $\mathbf{b}_s$ and $\mathbf{e}_s$.

Similar relationships are extracted for PAR or PARMA models of higher order. If the processes are not periodic (seasonal) but stationary, the same equations apply but in a simplified form as all parameter matrices do not depend on subperiod.

It is very common in stochastic hydrology the case that the lower-level model is expressed in terms of the logarithmic transformation of the lower-level variables, e.g., in terms of

$$\mathbf{X}_s^* := \ln(\mathbf{X}_s - \mathbf{c}_s) \tag{36}$$

where $\mathbf{c}_s$ is a vector of parameters estimated in such a manner that $\mathbf{X}_s^*$ be (approximately) normally distributed. In this case, relationships (31)-(35) express the covariances of the logarithmic transformations of variables. It is easy then to derive the covariances of the untransformed variables (which will be used then in the transformation) through the relation

$$\sigma_{sr}^{lj} = (\mu_s^l - c_s^l)\,(\mu_r^j - c_r^j)\,[\exp(\sigma_{sr}^{lj\,*}) - 1] \tag{37}$$

valid for any s, r, l and j, where $\sigma_{sr}^{lj} = \mathrm{Cov}[X_s^l, X_r^j]$, $\sigma_{sr}^{lj\,*} = \mathrm{Cov}[X_s^{l\,*}, X_r^{j\,*}]$, and

$$\mu_s^l = E[X_s^l] = c_s^l + \exp(\mu_s^{l\,*} + \sigma_{ss}^{ll\,*} / 2) \tag{38}$$

with $\mu_s^{l\,*} = E[X_s^{l\,*}]$.

In conclusion, any lagged covariance matrix of lower-level variables ($\boldsymbol{\sigma}_{sr}$) can be determined in terms of the lower-level model parameters by either of the two methods (options) proposed. The next step is to determine covariances between lower-level and higher-level variables ($\boldsymbol{\tau}_{sp}$). This is rather easy, because (1) implies that

$$\boldsymbol{\tau}_{sp} = \mathrm{Cov}[\mathbf{X}_s, \mathbf{Z}_p] = \sum_{r=(p-1)k+1}^{pk} \mathrm{Cov}[\mathbf{X}_s, \mathbf{X}_r] \tag{39}$$

or

$$\boldsymbol{\tau}_{sp} = \sum_{r=(p-1)k+1}^{pk} \boldsymbol{\sigma}_{sr} \tag{40}$$

What it remains is the determination of lagged covariances between higher-level variables ($\boldsymbol{\varphi}_{sp}$). Again using (1) we get

$$\boldsymbol{\varphi}_{pr} = \mathrm{Cov}[\mathbf{Z}_p, \mathbf{Z}_r] = \sum_{s=(p-1)k+1}^{pk} \mathrm{Cov}[\mathbf{X}_s, \mathbf{Z}_r] \tag{41}$$

or

$$\boldsymbol{\varphi}_{pr} = \sum_{r=(p-1)k+1}^{pk} \boldsymbol{\tau}_{sr} \tag{42}$$

We emphasize that the above estimation of $\boldsymbol{\varphi}_{pr}$ has been based on the lower-level model, although it could also be based on the higher-level model. However, in the latter case, possible incompatibilities of the two models would have negative consequences in preservation of the additive property. This is easily demonstrated through the simplest transformation (13): If φ_{11} is estimated from the lower-level model (equation (42)), i.e., as the sum of $\tau_{s\,1}$ for all s, then the coefficients $\tau_{s\,1} / \varphi_{11}$ add up to 1 and (13) preserves the additive property. On the contrary, if φ_{11} were estimated directly from the higher-level model, possibly it would have some departure from the sum of $\tau_{s\,1}$ (due to incompatibilities of models), which would result in violation of the additive property. This situation, that is, the estimation of higher-level covariances using the lower-level model, may seem strange at first glance. However, a more careful consideration of the context where these estimations of covariances are used shows that it is absolutely justified. Specifically, these estimations are not used in the higher-level model at all. On the contrary, this is an independent model that is fitted in a different manner (the appropriate one for the specific model chosen, which is out of the scope of this paper). Moreover, the higher-level model is run in an initial modeling phase, previous to that of the lower-level mode. In turn, the lower-level model is fitted with a procedure that is appropriate for this specific model, which again is out of the scope of this paper. Thus, the estimations of covariances described in the present section are used only in the coupling transformation, which applies to the values generated by the lower-level model. Therefore, it is natural to infer these parameters using the lower-level model only.

5 Preservation of skewness

The preservation of skewness is often of great importance, as hydrologic processes, particularly in small time scales, exhibit non-symmetric distributions. In all analyses of the previous sections, the skewness of the processes, either higher- or lower-level, was not considered. On the contrary, the analyses focused on second, marginal or joint, moments of the processes, the preservation of which was proved theoretically. Unfortunately, the preservation of third moments is hard to be handled in an analytical manner.

From the general relation (4), used to develop the various forms of the coupling transformation, we may conclude that the marginal third moments of $\mathbf{X}$ do not necessarily equal those of $\tilde{\mathbf{X}}$. Specifically, we may assume that, the marginal third moment of the term $\mathbf{h}\,(\mathbf{Y} - \tilde{\mathbf{Y}})$ is zero, due to symmetry, and thus it is not responsible for differences between the coefficients of skewness of $\tilde{\mathbf{X}}$ and $\mathbf{X}$. However, apart for that marginal moment, joint third moments of $\tilde{\mathbf{X}}$ and $\mathbf{h}\,\tilde{\mathbf{Y}}$ may create such differences. These joint third moments are difficult to determine analytically. Generally, because $\mathbf{X}$ in (4) is expressed as a linear combination of $\tilde{\mathbf{X}}$ and other variables, we expect that the coefficient of skewness of $\mathbf{X}$ will be lower than that of $\tilde{\mathbf{X}}$ (from the central limit theorem we know that, under certain conditions, linear combinations of variables tend to have symmetric distributions). Indeed, numerical investigations confirm this observation. Since an analytical solution is too complicated (if not intractable), we seek for approximate numerical methods. We will discuss two such methods.

Let $\zeta_s^l := E[(X_s^l - \mu_s^l)^3]$ and $\tilde{\zeta}_s^l := E[(\tilde{X}_s^l - \mu_s^l)^3]$ be the third central moments of X_s^l and $\tilde{X}_s^l$, respectively, where $\mu_s^l = E[X_s^l] = E[\tilde{X}_s^l]$. From the properties of the lower-level model we know ζ_s^l. In the first method we assume that $\tilde{\zeta}_s^l$ shall be different from (generally, higher than) ζ_s^l and we seek for the value of $\tilde{\zeta}_s^l$ that results in the correct value of ζ_s^l. This can be determined by iterative stochastic (Monte-Carlo) simulation. At the ith iteration we assume a trial value $(\tilde{\zeta}_s^l)_i$, starting with an initial value $(\tilde{\zeta}_s^l)_0 = \zeta_s^l$. We run the lower-level model to obtain a synthetic time series $\tilde{\mathbf{x}}_s$ with a sufficient length, and the coupling transformation to derive the series $\mathbf{x}_s$. From the latter we estimate the sample third moments which we denote $(\hat{\zeta}_s^l)_i$ (for location l, subperiod s and iteration i). We modify then $\tilde{\zeta}_s^l$ according to the rule

$$(\tilde{\zeta}_s^l)_{i+1} = (\tilde{\zeta}_s^l)_i + [\zeta_s^l - (\hat{\zeta}_s^l)_i] / c \tag{43}$$

and proceed to the next iteration. The denominator c in (43) is a number greater than 1 (e.g., $c = 2$) that enhances numerical stability in the route to the final solution. Normally, this procedure will stop when the attained sample third moments $(\hat{\zeta}_s^l)$ match the theoretical ones (ζ_s^l) for all l and s. However, given the Monte-Carlo character of the method, we must relax the convergence criterion and accept the solution of iteration i if for all l and s

$$|(\hat{\zeta}_s^l)_i - \zeta_s^l| \le \max_i \{(\hat{\tilde{\zeta}}_s^l)_i - (\tilde{\zeta}_s^l)_i|\} \tag{44}$$

where $\hat{\tilde{\zeta}}_s^l$ denotes the sample third moment of the synthetic time series $\tilde{\mathbf{x}}_s$. Practically, this means that a deviation of the sample skewness, after performing the coupling transformation, from its theoretical value can be acceptable if it is lower than or equal to the corresponding deviation without applying the coupling transformation.

The second method is based on conditional sampling in a manner much the same with that proposed by *Koutsoyiannis and Manetas* [1996]. Here we demand that the departure of $\tilde{\mathbf{Y}}$ and $\mathbf{Y}$ in the coupling transformation be small enough so that the addition of the term $\mathbf{h}\,(\mathbf{Y} - \tilde{\mathbf{Y}})$ to $\tilde{\mathbf{X}}$ do not affect the statistics of the latter. Therefore, we can assume that $\tilde{\zeta}^l_s = \zeta^l_s$. To achieve a vector $\tilde{\mathbf{Y}}$ close to the known $\mathbf{Y}$, we must keep repeating the generation process for the variables of each period (rather than performing a single generation only) until the distance of $\tilde{\mathbf{Y}}$ from $\mathbf{Y}$ be lower that an accepted limit. This distance can be defined as

$$\Delta = (1 / m) \, \|\mathbf{Y}' - \tilde{\mathbf{Y}}'\| \tag{45}$$

where $\mathbf{Y}'$ and $\tilde{\mathbf{Y}}'$ are $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ standardized by standard deviation (i.e. $Y'^l_s = Y^l_s / \{\mathrm{Var}[Y^l_s]\}^{1/2}$, $\tilde{Y}'^l_s = \tilde{Y}^l_s / \{\mathrm{Var}[Y^l_s]\}^{1/2}$), m is the common size of $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$, and $\|.\|$ denotes the Euclidian norm (other norms such as the maximum norm were found to behave worse).

Because in the initial generation scheme proposed, the variables $\tilde{\mathbf{X}}$ are generated independently of the higher-level variables, it was assumed in the Proposition of section 2 that $\mathbf{Y}$ is independent of $\tilde{\mathbf{X}}$. However, repetition apparently introduces dependence of $\tilde{\mathbf{X}}$ on $\mathbf{Y}$. Hence, the question arises whether the conclusions of the Proposition are still valid for $\mathbf{Y}$ dependent on $\tilde{\mathbf{X}}$ (and $\tilde{\mathbf{Y}}$) or not. For an intuitive answer to that question we observe that the case of independent $\mathbf{Y}$ and $\tilde{\mathbf{X}}$ is the worst to manage. If the covariance $\mathrm{Cov}[\tilde{\mathbf{X}}, \mathbf{Y}]$ approaches (or matches) the true covariance $\mathrm{Cov}[\mathbf{X}, \mathbf{Y}]$, instead of being zero, then it is easier for the coupling transformation to preserve the statistical properties of interest. Furthermore, the applications given by *Koutsoyiannis and Manetas* [1996] for the similar case of using their adjustment procedure, which is equivalent to the simplified transformation of subsection 3.1, and those given here in section 6 below, verify empirically a positive answer to the above question.

That the answer is positive, under certain conditions, can be proved theoretically. Specifically, it is shown in Appendix A2 that under the assumption that $\mathbf{Y}$ is no more independent from $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ but correlated to both, such that

$$\mathrm{Cov}[\tilde{\mathbf{X}}, \mathbf{Y}] = \mathbf{h}\,\mathrm{Cov}[\tilde{\mathbf{Y}}, \mathbf{Y}] \tag{46}$$

where $\mathbf{h}$ is given from (5), the Proposition of section 2 remains valid in all its items. We note that the condition (46) holds in the case of $\mathbf{Y}$ independent from $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$, as both its sides are zero. Also it holds in the other extreme case where both $\mathrm{Cov}[\tilde{\mathbf{X}}, \mathbf{Y}]$ and $\mathrm{Cov}[\tilde{\mathbf{Y}}, \mathbf{Y}]$ match the true covariances $\mathrm{Cov}[\mathbf{X}, \mathbf{Y}]$ and $\mathrm{Cov}[\mathbf{Y}, \mathbf{Y}]$, respectively. This can be verified using (5). Numerical investigation with the proposed repetition scheme shows that the condition holds in intermediate cases as well.

Both the above methods can lead to sufficient preservation of skewness (see section 6 below), although none of them is ideal. Their common disadvantages are the approximate character and the repetitive application, which increases computer time (although this time is not an obstacle as it ranges from less than a minute to some minutes for typical hydrological problems run on a modern PC to generate some thousands of synthetic data). Their common advantages are their simplicity and independence of the type of models. The Monte-Carlo simulation method seems to need fewer repetitions that the conditional sampling method and, once its additional parameters ($\tilde{\zeta}_s^l$) are evaluated, it does not need further repetitions in subsequent applications of the model. However, to get adequate estimates of these parameters a large length of simulation record (e.g., 10 000 years or more) is needed. Another weak point of the Monte-Carlo method is the fact that it results in coefficients of skewness higher than those of the original lower-level model, and this may be a problem if the latter are already large. Generally, we can consider the Monte-Carlo method preferable if a model must be setup once and thereafter run several times. Conversely, if the model is to run only once, the conditional sampling method may be preferable.

Apart from the computational cost, i.e., the increase of computer time due to repetition, no additional cost is implied by either of the two approaches for preservation of skewness. Specifically, there is no negative effect in preserving other properties of the lower-level processes such as the second order moments. On the contrary, the conditional sampling method may have positive effects in preserving second order moments by simplified forms of the coupling transformation, as it is demonstrated in section 6.

6 Performance demonstration

The entire modelling framework for both time scales can be summarized in the following steps composing two groups. The steps of the first group correspond to model choice and fitting:

1. Choose a model for the higher-level (coarse) time scale and fit it using the appropriate method for that model.
2. Choose a model for the lower-level (fine) time scale and fit it using the appropriate method for that model.
3. Decide the composition of the vector $\mathbf{Y}$ according to the specific needs of the problem as discussed in section 3. In the most general multivariate case use the composition defined in (22).
4. Define the appropriate matrix $\mathbf{h}$ of the coupling transformation using the corresponding equations of section 3. In the most general multivariate case use (25).
5. Evaluate the items of matrix $\mathbf{h}$ using either of the methods of section 4.

The steps of the second group perform the generation:

6. Use the higher-level model to produce a series of $\mathbf{Z}$ with the desired length.
7. Use the lower-level model to produce a series of $\tilde{\mathbf{X}}$ with the same length, without reference to the higher-level series.

8. At each period evaluate the vectors **Y** and $\widetilde{\mathbf{Y}}$ using the values of **Z,** $\widetilde{\mathbf{X}}$ of the current and (if applicable) next period, and (if applicable) **X** of the previous period.
9. Apply the coupling transformation to derive **X** of the current period.
10. Repeat steps 8 and 9 for all periods.

For the preservation of skewness the algorithm becomes slightly more complex due to repetition as described in section 5.

The different forms and methods of the proposed framework for coupling stochastic models of different time scales are demonstrated through a simple numerical example involving two locations and two lower-level variables per period. The higher- and lower-level models used are those of equations (2) and (3), respectively (we remind that models (2) and (3) are not fully compatible one another). Several sets of model parameters were examined; here we present the results of a representative case with the parameter set shown in Table 1.

The following forms of the coupling transformation were examined:

1. Full transformation, multivariate mode; as in subsection 3.4 (symbolically: F/M).
2. Full coupling transformation, single variate mode, i.e., transformation applied separately to each location; as in subsection 3.3 (symbolically: F/S).
3. Transformation without link to the higher-level variable of the next period, as in subsection 3.2, but in multivariate mode (symbolically: N+/M).
4. Transformation without link to the lower-level variables of the previous period, multivariate mode (symbolically: N–/M).
5. Simplified transformation, single variate mode; as in subsection 3.1 (symbolically: S/S).
6. Modified simplified transformation, single variate mode (symbolically: S1/S).

The modification in item 6 of the above list (in comparison to item 5) consists of using the value of the last lower-level variable of the previous period for initialization the PAR(1) model (3) in each period, although this is not used by the coupling transformation.

For comparison, results of the non-coupled lower-level model (symbolically: NC) are also presented. In all cases the model generated synthetic series of 10 000 periods, from which the sample statistics were computed and compared to the theoretical values.

In Figure 2 we compare the marginal statistics (means, standard deviations and coefficients of skewness) of all lower-level variables, obtained by the different transformation forms, to their theoretical values. As anticipated, all forms of coupled models preserved perfectly the means and standard deviations but no form preserved the coefficients of skewness (apart, of course, from the non-coupled model). Figure 3 shows the temporal correlation coefficients of the lower-level variables with previous lower-level variables and current and next higher-level variables, as derived by the various forms of the coupling transformation for the test application. We observe that only the full transformation form, either in multivariate or single-variate mode (F/M, F/S) has a perfect behavior in preserving all these correlations. Transformation form N+/M fails to reproduce some of the correlations with higher-level variables of next period; also, it has a lower performance in preserving correlations with previous lower-level variables. Transformation forms N−/M and S/S exhibit a poor behavior in preserving correlations with previous lower-level variables (particularly, those of previous period); the situation is improved with model S1/M. Also, both simplified models (S/S, S1/S) fail to reproduce the correlations with next higher-level variables. Figure 4 shows the lag-zero cross-correlation coefficients attained by the various transformation forms. As anticipated, the multivariate forms (F/M, N+/M, N–/M) performed very well whereas single-variate models (F/S, S/S, S1/S) failed to preserve cross-correlations.

To improve the preservation of the coefficients of skewness we applied both methods discussed in section 5. In Figure 5 we present the results of the Monte-Carlo method for the full transformation form in multivariate mode (F/M). We observe that after the tenth iteration, the attained coefficients of skewness become close to the theoretical ones. The criterion of equation (44) results true for iteration 13; the fluctuation of the attained coefficients of skewness of most variables that appears beyond iteration 13 are anticipated because of the Monte-Carlo character of the method. We notice that the differences of the assumed and attained (after applying the coupling transformation) coefficients of skewness, which correspond to $\tilde{\zeta}_s^l$ and $\hat{\zeta}_s^l$, respectively, may be very large (e.g., for variable X_1^1).

We also applied the method of conditional sampling using repetitions for the full (F/M), the simplified (S/S) and the modified simplified (S1/S) forms of the coupling transformation. In Figure 6 we plotted the average number of repetitions required to achieve a certain distance Δ (defined in (45)). Figure 7 shows the attained coefficients of skewness using the conditional sampling method, as a function of the mean number of repetitions. We observe that all three transformation forms examined have roughly the same performance. Adequate values of sample coefficients of skewness are obtained with 50-100 repetitions. We also examined in this case the preservation of correlation coefficients of the lower-level variables with the previous lower-level variables and the next higher-level variables (Figure 8) and cross-correlation coefficients (Figure 9). We observe that both S/S and S1/S forms, which failed to preserve all these statistics if applied without repetitions (Figure 3, Figure 4), result in adequate preservation of cross-correlations after 50-100 repetitions. In addition, the S1/S model performs well in preserving correlation coefficients of the lower-level variables with the previous lower-level variables after 50-100 repetitions. However, none of the two simplified forms could approach the theoretical correlation coefficients of the lower-level variables with the next higher-level variables, even after 1000 repetitions. In conclusion, repetition, apart from its usefulness for preserving coefficients of skewness, improves also preservation of auto- and cross-correlation coefficients of lower-level variables of simplified model versions. Notably, this is done at no additional computational cost.

7 Summary and conclusions

A methodology is proposed for coupling stochastic models of hydrologic processes applying to different time scales so that time series generated by the different models be consistent. Given two multivariate time series, generated by two separate (unrelated) stochastic models of the same hydrologic process, each applying to a different time scale, a transformation is developed (referred to as a coupling transformation) that appropriately modifies the time series of the lower-level time scale so that this series be consistent with the time series of the higher-level time scale without affecting the second-order stochastic structure of the former and also establishing appropriate correlations between the two time series. The coupling transformation is based on a developed generalized mathematical proposition, which ensures preservation of marginal and joint second-order statistics and linear relationships between lower- and higher-level processes. The methodology can be applied to problems involving disaggregation of annual to seasonal and seasonal to subseasonal time scales, as well as problems involving finer time scales, under the only requirement that a specific stochastic model is available for each involved time scale. An implementation of the methodology for disaggregation of daily rainfall into hourly rainfall at many locations (a problem much more demanding than disaggregation of annual to seasonal quantities, due to the intermittent aspect of the process and the very asymmetric marginal distributions) is under way.

Several specific forms of the coupling transformation are studied. The simplest of them, symbolically S/S and S1/S, are single variate and do not consider any link to higher- or lower-level variables of previous or next periods; the difference of the two is that S1/S uses some of the already generated variables of the previous period for its initialization whereas S/S does not. The most detailed form, symbolically F/M, is multivariate and incorporates appropriate links to higher- and lower-level variables of previous and next periods. In addition, techniques for evaluating parameters of the coupling transformation based on second order moments of the lower-level process are studied. Specific implementations of these techniques are given for the very common cases where the lower-level process (or its logarithmic transformation) is multivariate PAR(1), PAR(2) or PARMA(1, 1).

Although the coupling transformation can explicitly preserve means and second-order statistics of the processes involved, it introduces bias to the coefficients of skewness and any other parameters that cannot be related to means and second order statistics (e.g. probabilities of dry intervals in the case of the fine-scale rainfall process). Due to its linearity, the coupling transformation encompasses the effects of the central limit theorem. Thus, the transformed series tend to be Gaussian (their coefficients of skewness are reduced). Unlike second-order statistics, third moments and coefficients of skewness are too complicated to handle analytically. However, two approximate methods that enable preservation of skewness of the processes are studied. The first introduces negative bias to the coefficients of skewness of the lower-level processes, the magnitude of which is determined by Monte-Carlo simulation, to counterbalance the bias introduced by the application of the coupling transformation. The second uses repetition as a means of conditional sampling and, in that way, it prevents the lower-level variables from departing (in terms of their sum) significantly from the known higher-level variables, thus reducing bias to a negligible level.

A detailed numerical example of the application of the methodology demonstrates that it behaves as it should. The full multivariate (F/M) form preserves all temporal and spatial correlations of lower-level variables either with other lower-level variables or with higher-level variables whereas simplified forms fail to preserve some of these correlations. All forms preserve first and second marginal moments but fail to preserve third moments. The latter are preserved only after application of either of the two methods developed for that purpose, Monte-Carlo simulation or conditional sampling. The latter, apart from its usefulness for preservation of skewness coefficient, improves also (at no additional computational cost) preservation of auto- and cross-correlation coefficients of lower-level variables for simplified forms of the coupling transformation.

Among the different forms of the coupling transformation studied, the full multivariate one (F/M) is the most preferable as it preserves explicitly the greater number of statistics. Between the two methods for preserving skewness, the Monte-Carlo method may be preferable if a model must be setup once and thereafter run several times. Conversely, if the model is to run only once, the conditional sampling method may be preferable. Besides, if a simplified form of the coupling transformation is chosen, then it must be combined with the conditional sampling method to improve preservation of statistics that are not explicitly considered in the transformation.

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Table 1 Model parameters for the test application

Parameter type		Parameter values			
		Lower-level process			
		Subperiod s = 1		*Subperiod s = 2*	
		Location $l = 1$	Location $l = 2$	Location $l = 1$	Location $l = 2$
	Variable symbol	X_1^1	X_1^2	X_2^1	X_2^2
Mean, μ_s^l		1.000	2.000	3.000	4.000
Covariance matrices					
$\boldsymbol{\sigma}_{ss}$	$l = 1$	0.250	0.210	0.810	0.432
	$l = 2$	0.210	0.490	0.432	2.560
$\boldsymbol{\sigma}_{s,s-1}$	$l = 1$	0.225	0.120	0.090	0.076
	$l = 2$	0.113	0.672	0.432	1.008
Third central moment, ζ_s^l		0.125	0.240	0.437	6.550
		Higher-level process			
		Location $l = 1$	Location $l = 2$		
	Variable symbol	Z_1^1	Z_1^2		
Mean		4.000	6.000		
Covariance matrices					
$\boldsymbol{\varphi}_{11}$	$l = 1$	1.240	1.150		
	$l = 2$	1.150	5.066		
$\boldsymbol{\varphi}_{12}$	$l = 1$	0.340	0.693		
	$l = 2$	0.192	2.863		
Third central moment		0.708	10.704		

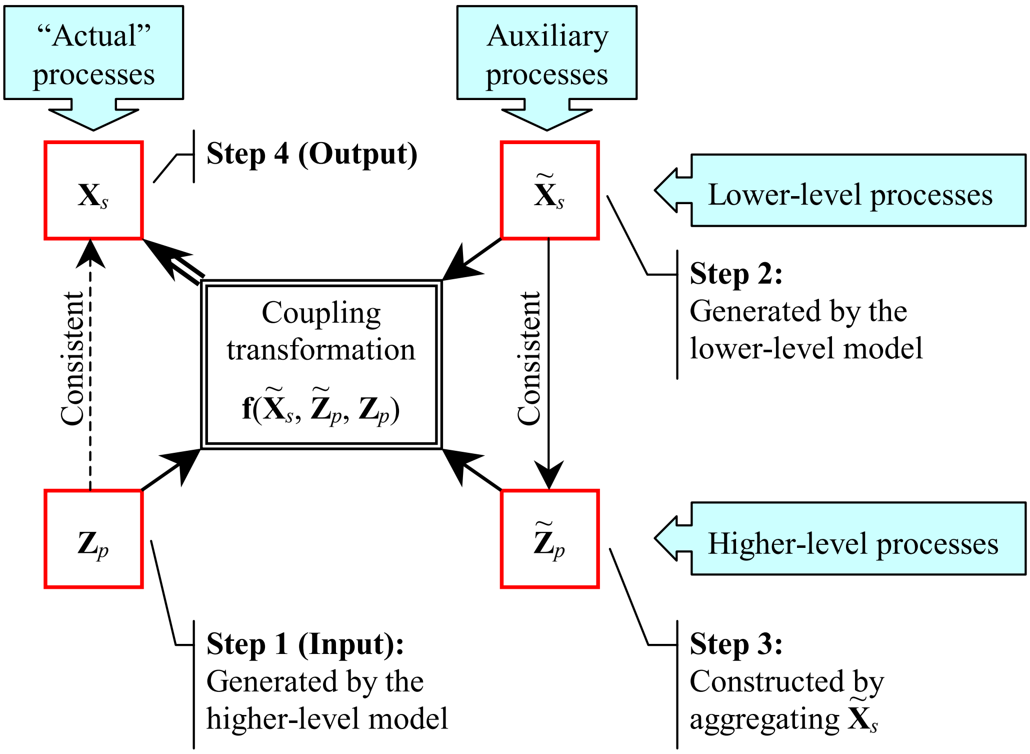

Figure 1 Schematic representation of actual and auxiliary processes, their links, and the steps followed to construct the actual lower-level process from the actual higher-level process.

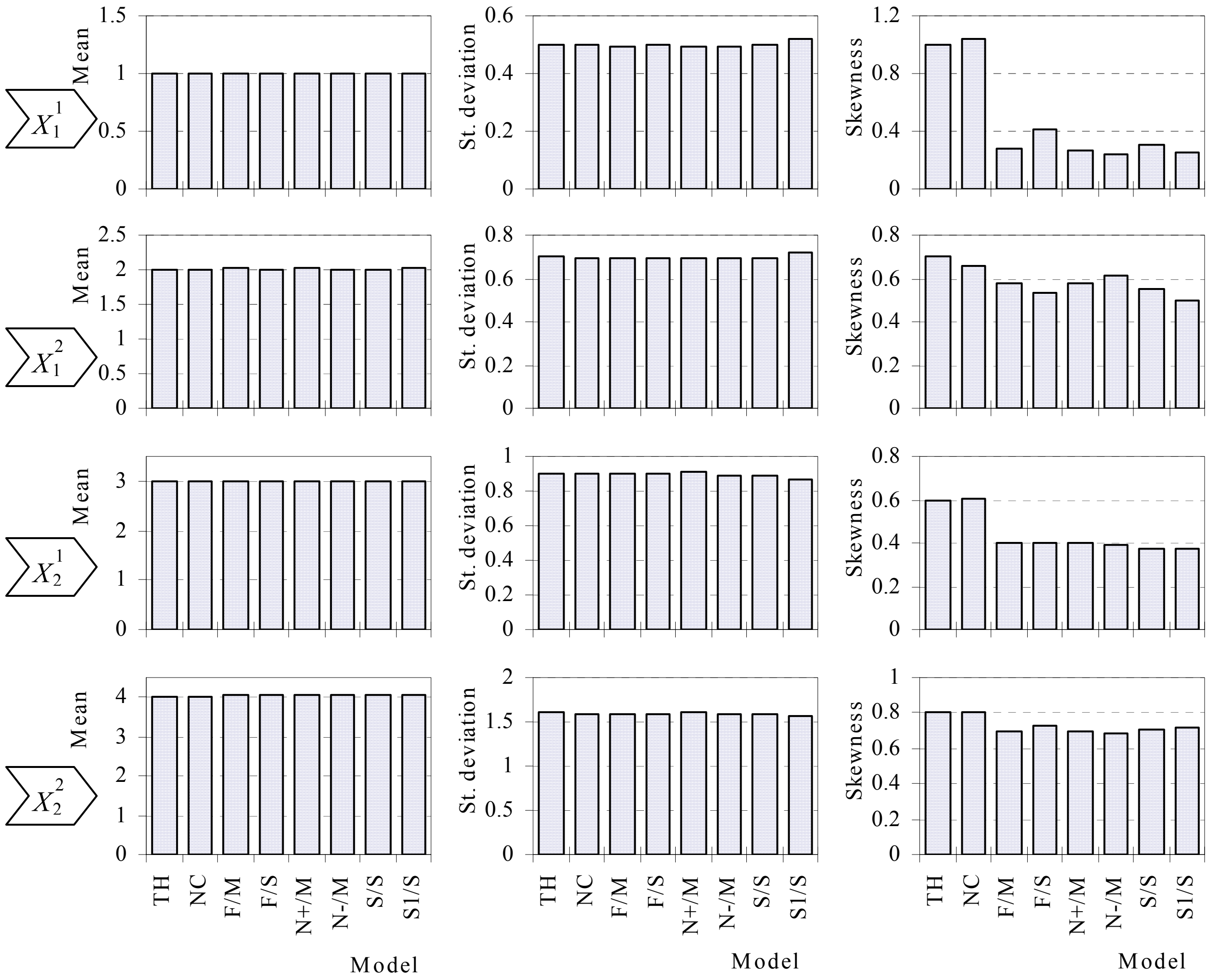

Figure 2 Comparison of marginal statistics of the lower-level variables (X_s^l, indicated by the block arrows to the left) as derived by various forms of the coupling transformation for the test application. Key: TH: Theoretical values; NC: Non-coupled lower-level model; F/M: Full coupling transformation, multivariate mode; F/S: Full transformation, single variate mode; N+/M: Transformation without link to the next higher-level variable, multivariate mode; N−/M: Transformation without link to the previous lower-level variable, multivariate mode; S/S: simplified transformation, single variate mode; S1/S: modified simplified transformation (starting with the known value of the previous lower-level variable), single variate mode.

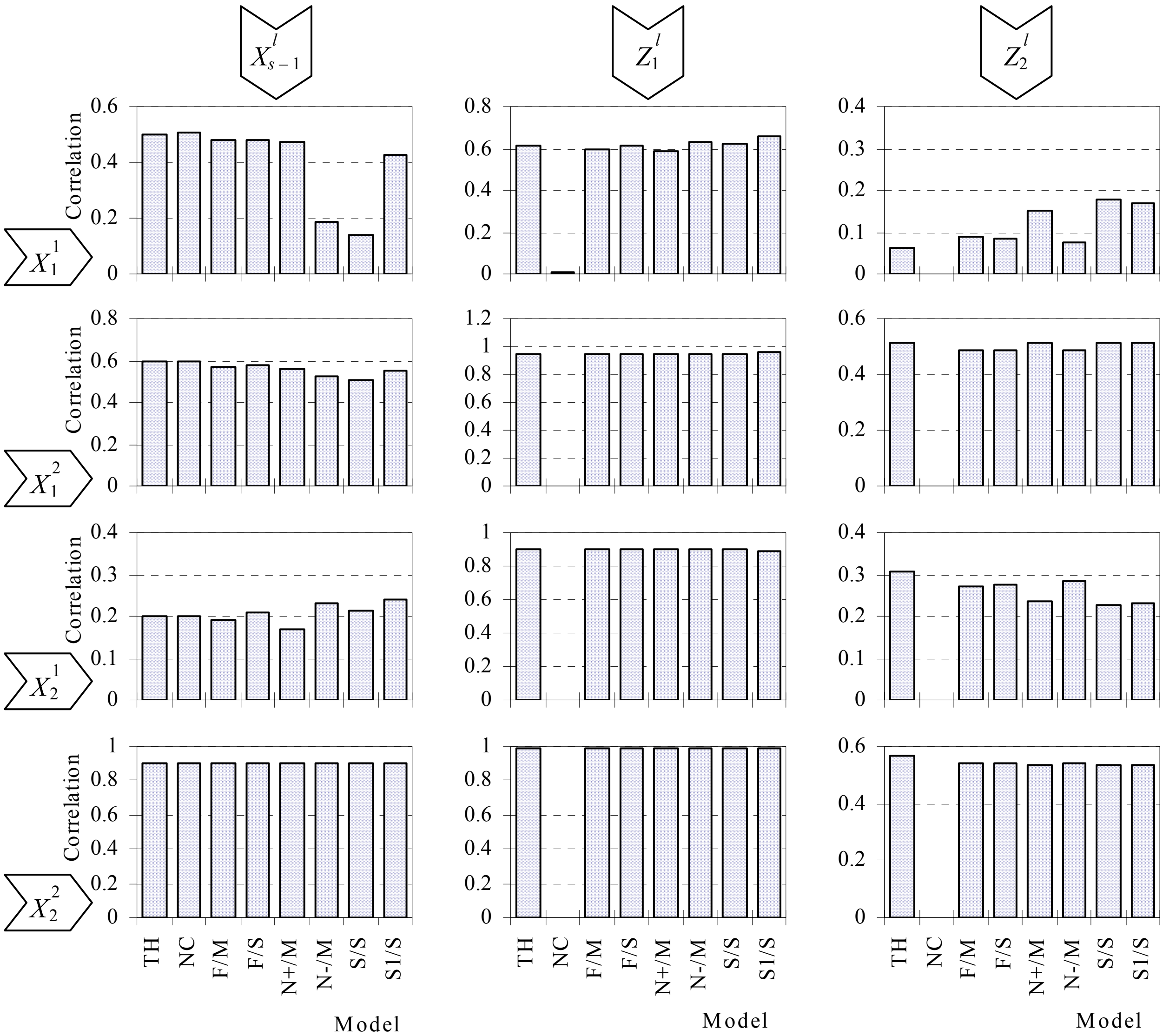

Figure 3 Comparison of temporal correlation coefficients of the lower-level variables (X^l_s, indicated by the block arrows to the left) with previous lower-level variables and current and next higher-level variables (X^l_{s-1}, Z^l_1, Z^l_2, respectively, indicated by the block arrows in the top) as derived by various forms of the coupling transformation for the test application. Key: same as in Figure 2.

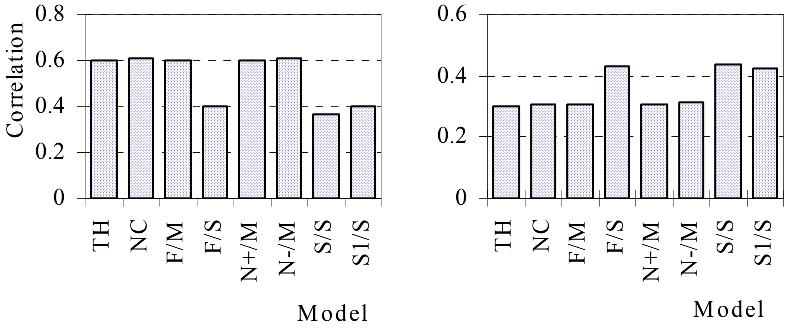

Figure 4 Comparison of cross-correlation coefficients of the lower-level variables of the first (left) and second (right) subperiod as derived by various forms of the coupling transformation for the test application. Key: same as in Figure 2.

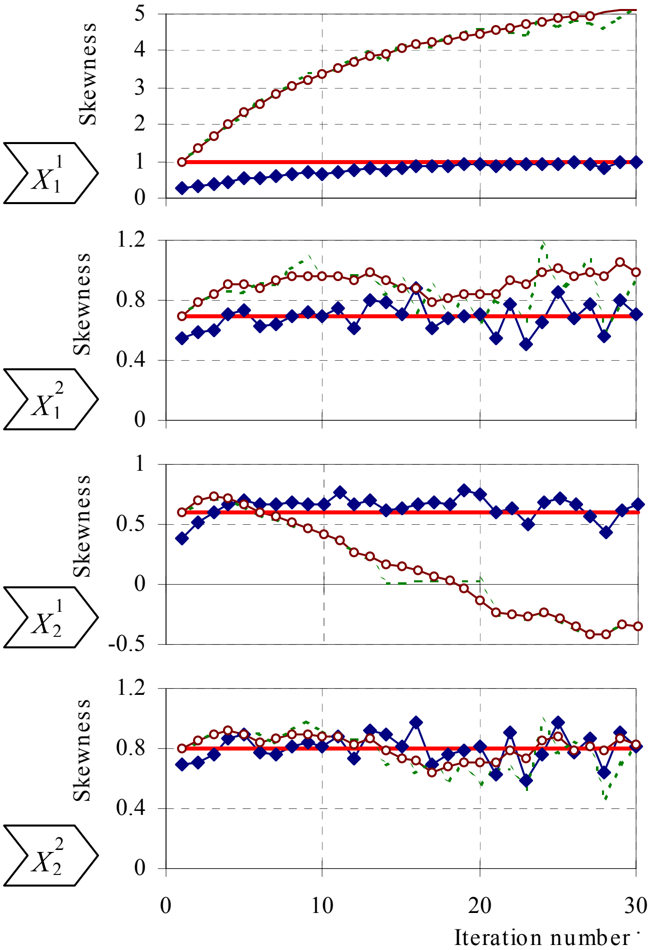

Figure 5 Hypothesized (open circles, corresponding to $\tilde{\zeta}^l_s$) and attained (diamonds, corresponding to $\hat{\zeta}^l_s$) coefficients of skewness for each of the lower-level variables (shown in the block arrows to the left), as a function of the iteration number, for a test application of the full coupling transformation (F/M, Monte-Carlo method). Full and dotted lines represent the theoretical values (corresponding to ζ^l_s) and the values obtained from the non-coupled lower-level model (NC, corresponding to $\tilde{\zeta}^l_s$), respectively.

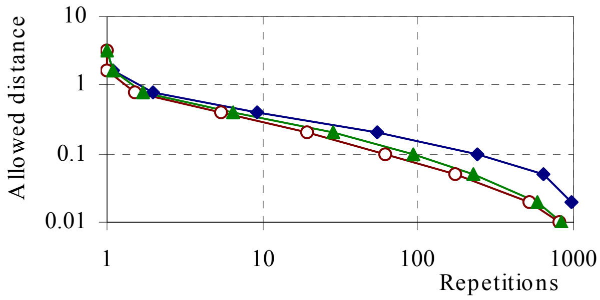

Figure 6 Average number of repetitions required to achieve the preset allowed distance between **Y** and $\tilde{\mathbf{Y}}$ using the conditional sampling method for the full (F/M, diamonds), the simplified (S/S, triangles), and the modified simplified (S1/S, open circles) coupling transformations.

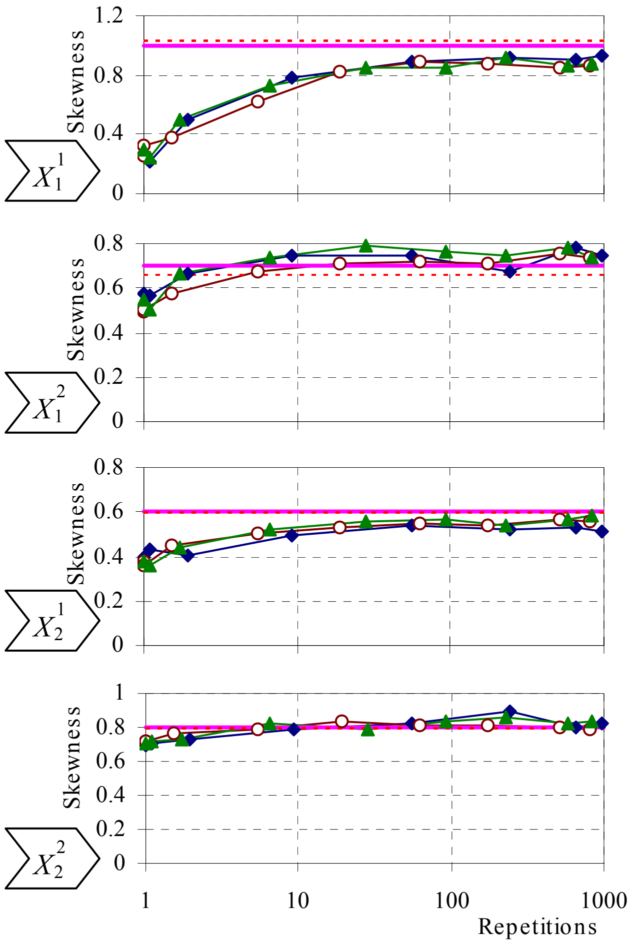

Figure 7 Attained coefficients of skewness of the lower-level variables (X_s^l, indicated by the block arrows to the left) as a function of the mean number of repetitions, for the full (F/M, diamonds), the simplified (S/S, triangles), and the modified simplified (S1/S, open circles) coupling transformations. Full and dotted lines represent the theoretical values and the values obtained from the non-coupled lower-level model (NC), respectively.

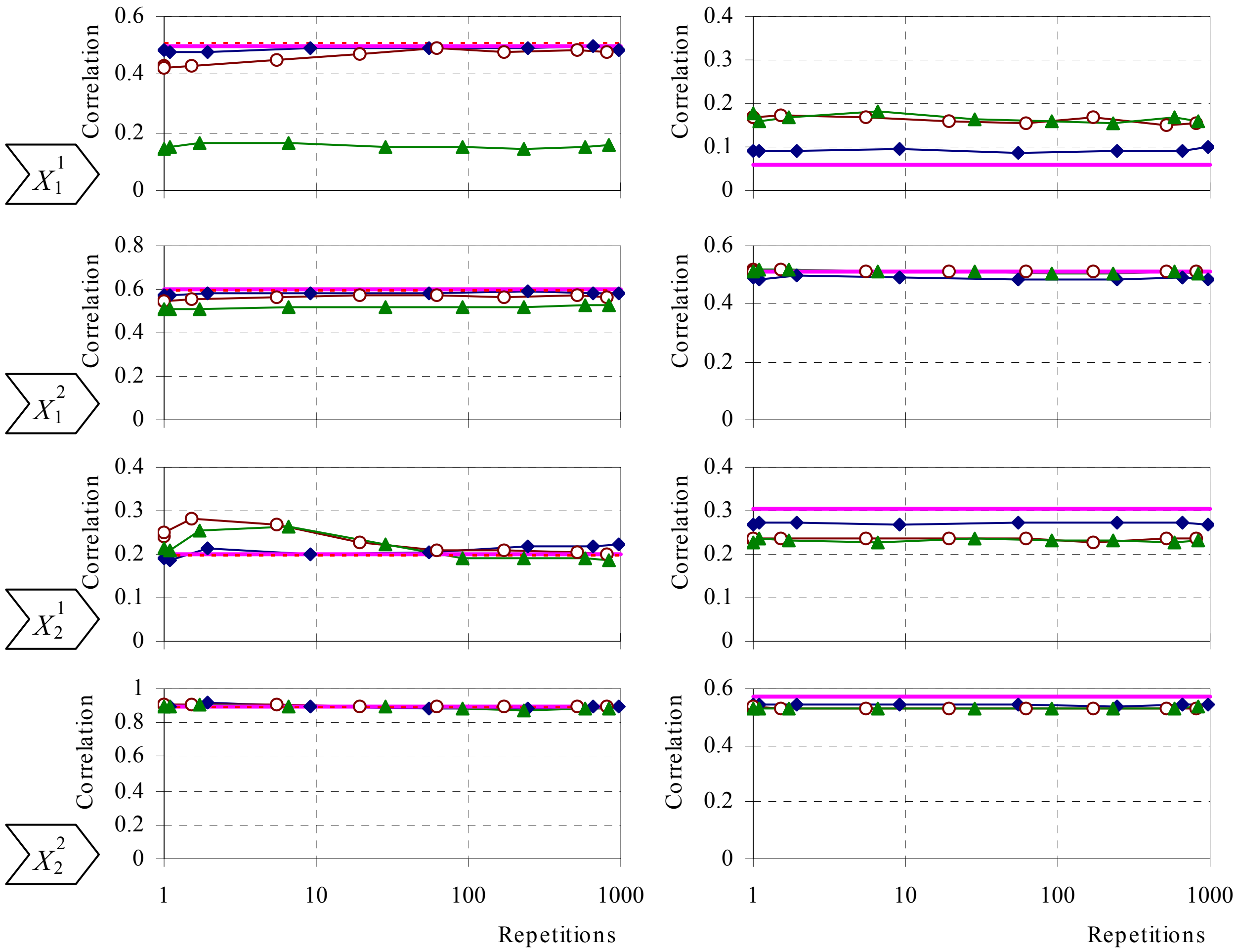

Figure 8 Attained correlation coefficients of the lower-level variables (X_s^l, indicated by the block arrows to the left) with the previous lower-level variables (left column) and the next higher-level variables (right column) as a function of the mean number of repetitions, for the full (F/M, diamonds), the simplified (S/S, triangles), and the modified simplified (S1/S, open circles) coupling transformations. Full and dotted lines represent the theoretical values and the values obtained from the non-coupled lower-level model (NC), respectively.

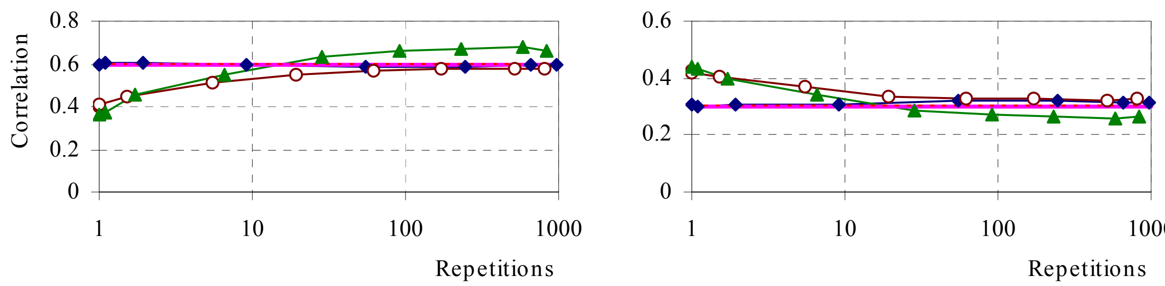

Figure 9 Attained cross-correlation coefficients of the lower-level variables of the first (left) and second (right) subperiod as a function of the mean number of repetitions, for the full (F/M, diamonds), the simplified (S/S, triangles), and the modified simplified (S1/S, open circles) coupling transformations. Full and dotted lines represent the theoretical values and the values obtained from the non-coupled lower-level model (NC), respectively.

Coupling stochastic models of different time scales

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Appendix – Mathematical derivations (Supplement on microfiche)

A1 Proof of the Proposition of section 2

(a) Firstly, we will prove that (4) preserves means and covariance matrices. Taking average values in both sides of (4) we find that $E[\mathbf{X}] = E[\tilde{\mathbf{X}}]$ (because by definition of $\mathbf{Y}$, $E[\mathbf{Y}] = E[\tilde{\mathbf{Y}}]$), which proves preservation of means. Subtracting means from both sides of (4) we get

$$(\mathbf{X} - E[\mathbf{X}]) = (\tilde{\mathbf{X}} - E[\tilde{\mathbf{X}}]) + \mathbf{h}\ \{(\mathbf{Y} - E[\mathbf{Y}]) - (\tilde{\mathbf{Y}} - E[\tilde{\mathbf{Y}}])\} \tag{A1}$$

Postmultiplying (A1) by its transpose and then taking expected values we get

$$\begin{aligned}\mathrm{Cov}[\mathbf{X}, \mathbf{X}] = {} & \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{X}}] - \mathrm{Cov}[\tilde{\mathbf{X}}, \mathbf{h}\,\tilde{\mathbf{Y}}] - \mathrm{Cov}[\mathbf{h}\,\tilde{\mathbf{Y}}, \tilde{\mathbf{X}}] \\ & + \mathrm{Cov}[\mathbf{h}\,\tilde{\mathbf{Y}}, \mathbf{h}\,\tilde{\mathbf{Y}}] + \mathrm{Cov}[\mathbf{h}\,\mathbf{Y}, \mathbf{h}\,\mathbf{Y}]\end{aligned} \tag{A2}$$

where we have omitted covariance terms among $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ or $\tilde{\mathbf{X}}$, because $\mathbf{Y}$ is independent of both $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$. Observing that by definition $\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}] = \mathrm{Cov}[\mathbf{Y}, \mathbf{Y}]$, we can write (A2) as

$$\mathrm{Cov}[\mathbf{X}, \mathbf{X}] = \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{X}}] - \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]\ \mathbf{h}^T - \mathbf{h}\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{X}}] + 2\ \mathbf{h}\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\ \mathbf{h}^T \tag{A3}$$

and using the definition of $\mathbf{h} := \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]\ \{\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\}^{-1}$ and observing that $\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]$ is symmetric, we get

$$\begin{aligned}\mathrm{Cov}[\mathbf{X}, \mathbf{X}] = {} & \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{X}}] - \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]\ \{\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\}^{-1}\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{X}}] \\ & - \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]\ \{\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\}^{-1}\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{X}}] \\ & + 2\ \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]\ \{\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\}^{-1}\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\ \{\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\}^{-1}\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{X}}]\end{aligned} \tag{A4}$$

which reduces to

$$\mathrm{Cov}[\mathbf{X}, \mathbf{X}] = \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{X}}] \tag{A5}$$

This proves our claim about preservation of covariances.

Postmultiplying (A1) by the $(\mathbf{Y} - E[\mathbf{Y}])^T$ and then taking expected values and also omitting covariance terms among $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ or $\tilde{\mathbf{X}}$, which are zero because $\mathbf{Y}$ is independent of both $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$, we get

$$\mathrm{Cov}[\mathbf{X}, \mathbf{Y}] = \mathrm{Cov}[\mathbf{h}\ \mathbf{Y}, \mathbf{Y}] = \mathbf{h}\ \mathrm{Cov}[\mathbf{Y}, \mathbf{Y}] \tag{A6}$$

Substituting **h** from (5) we get

$$\mathrm{Cov}[\mathbf{X}, \mathbf{Y}] = \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]\ \{\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\}^{-1}\ \mathrm{Cov}[\mathbf{Y}, \mathbf{Y}] = \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}] \tag{A7}$$

since $\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}] = \mathrm{Cov}[\mathbf{Y}, \mathbf{Y}]$. This proves our claim that the covariance matrix of **X** and **Y** is identical to that of $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$.

(b) Secondly, we will prove (7) assuming that (6) holds. Taking expected values in (6) and then subtracting from (6) we get

$$\mathbf{g}_{\mathbf{X}}^T\ \{\tilde{\mathbf{X}} - E[\tilde{\mathbf{X}}]\} = \mathbf{g}_{\mathbf{Y}}^T\ \{\tilde{\mathbf{Y}} - E[\tilde{\mathbf{Y}}]\} \tag{A8}$$

Postmultiplying (A8) by $\{\tilde{\mathbf{Y}} - E[\tilde{\mathbf{Y}}]\}^T$ and then taking expected values we find

$$\mathbf{g}_{\mathbf{X}}^T\ \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}] = \mathbf{g}_{\mathbf{Y}}^T\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}] \tag{A9}$$

Postmultiplying (A9) by $\{\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\}^{-1}$ we get

$$\mathbf{g}_{\mathbf{Y}}^T = \mathbf{g}_{\mathbf{X}}^T\ \mathbf{h} \tag{A10}$$

Multiplying both sides of (4) on the left by $\mathbf{g}_{\mathbf{X}}^T$ we find that

$$\mathbf{g}_{\mathbf{X}}^T\ \mathbf{X} := \mathbf{g}_{\mathbf{X}}^T\ \tilde{\mathbf{X}} + \mathbf{g}_{\mathbf{X}}^T\ \mathbf{h}\ (\mathbf{Y} - \tilde{\mathbf{Y}}) \tag{A11}$$

or

$$\mathbf{g}_{\mathbf{X}}^T\ \mathbf{X} := \mathbf{g}_{\mathbf{X}}^T\ \tilde{\mathbf{X}} + \mathbf{g}_{\mathbf{Y}}^T\ (\mathbf{Y} - \tilde{\mathbf{Y}}) = \mathbf{g}_{\mathbf{X}}^T\ \tilde{\mathbf{X}} + \mathbf{g}_{\mathbf{Y}}^T\ \mathbf{Y} - \mathbf{g}_{\mathbf{Y}}^T\ \tilde{\mathbf{Y}} \tag{A12}$$

which, given (6), results directly in (7).

(c) If we get covariances as in (A2) above, but conditionally on $\mathbf{Y} = \mathbf{y}$, the last term $\mathrm{Cov}[\mathbf{h}\ \mathbf{Y}, \mathbf{h}\ \mathbf{Y} \mid \mathbf{Y} = \mathbf{y}]$ will now be zero. The other terms are not affected by the condition because of independence from **Y**. Thus, writing (A2) for $\mathbf{Y} = \mathbf{y}$, we get

$$\mathrm{Cov}[\mathbf{X}, \mathbf{X} \mid \mathbf{Y} = \mathbf{y}] = \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{X}}] - \mathrm{Cov}[\tilde{\mathbf{X}}, \mathbf{h}\ \tilde{\mathbf{Y}}] - \mathrm{Cov}[\mathbf{h}\ \tilde{\mathbf{Y}}, \tilde{\mathbf{X}}] + \mathrm{Cov}[\mathbf{h}\ \tilde{\mathbf{Y}}, \mathbf{h}\ \tilde{\mathbf{Y}}] \tag{A13}$$

where we have omitted covariance terms among **Y** and $\tilde{\mathbf{Y}}$ or $\tilde{\mathbf{X}}$, because **Y** is independent of both $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$. Equivalently,

$$\mathrm{Cov}[\mathbf{X}, \mathbf{X} \mid \mathbf{Y} = \mathbf{y}] = \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{X}}] - \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]\ \mathbf{h}^T - \mathbf{h}\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{X}}] + \mathbf{h}\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\ \mathbf{h}^T \tag{A14}$$

In a similar manner as previously, this becomes

$$\mathrm{Cov}[\mathbf{X}, \mathbf{X} \mid \mathbf{Y} = \mathbf{y}] = \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{X}}] - \mathrm{Cov}[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]\ \{\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{Y}}]\}^{-1}\ \mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{\mathbf{X}}] \tag{A15}$$

which if written for the (i, i)th element of $\mathrm{Cov}[\mathbf{X}, \mathbf{X} \mid \mathbf{Y} = \mathbf{y}]$ takes the form of (8).

Next, we will show that $\mathrm{Var}[X_i \mid \mathbf{Y} = \mathbf{y}]$ is identical to the least mean square prediction error of X_i from **Y**. To this aim, we consider the linear prediction model

$$X_i = \boldsymbol{\kappa}^T\ \mathbf{Y} + U \tag{A16}$$

where $\boldsymbol{\kappa}$ is a vector of parameters and U is a random variable whose deviation from mean represents the prediction error. We seek for the vector $\boldsymbol{\kappa}$ that minimizes Var[U]. Taking expected values in both sides of (A16) and subtracting from (A16) we get

$$(U - E[U]) = (X_i - E[X_i]) - \boldsymbol{\kappa}^T \{\mathbf{Y} - E[\mathbf{Y}]\} \tag{A17}$$

so that

$$\mathrm{Var}[U] = \mathrm{Var}[X_i] - 2\,\mathrm{Cov}[X_i, \boldsymbol{\kappa}^T \mathbf{Y}] + \mathrm{Var}[\boldsymbol{\kappa}^T \mathbf{Y}] \tag{A18}$$

or equivalently,

$$\mathrm{Var}[U] = \mathrm{Var}[X_i] - 2\,\mathrm{Cov}[X_i, \mathbf{Y}]\,\boldsymbol{\kappa} + \boldsymbol{\kappa}^T\,\mathrm{Cov}[\mathbf{Y}, \mathbf{Y}]\,\boldsymbol{\kappa} \tag{A19}$$

To find $\boldsymbol{\kappa}$ that minimizes Var[U] we take the derivative of the right-hand side of (A19) with respect to $\boldsymbol{\kappa}$ and equate it to 0. This results in

$$-\,2\,\mathrm{Cov}[X_i, \mathbf{Y}] + 2\,\boldsymbol{\kappa}^T\,\mathrm{Cov}[\mathbf{Y}, \mathbf{Y}] = \mathbf{0} \tag{A20}$$

or

$$\boldsymbol{\kappa} = \{\mathrm{Cov}[\mathbf{Y}, \mathbf{Y}]\}^{-1}\,\mathrm{Cov}[\tilde{\mathbf{Y}}, \tilde{X}_i] \tag{A21}$$

Substituting this result in (A19) we get

$$\mathrm{Var}[U] = \mathrm{Var}[X_i] - \mathrm{Cov}[X_i, \mathbf{Y}]\,\{\mathrm{Cov}[\mathbf{Y}, \mathbf{Y}]\}^{-1}\,\mathrm{Cov}[\mathbf{Y}, X_i] \tag{A22}$$

Taking into consideration item (a) of the Proposition we conclude that Var[U] is identical to Var[X_i | $\mathbf{Y} = \mathbf{y}$] given by (8).

A2 Extension of the Proposition for auxiliary processes dependent on the actual higher-level process

If Cov[$\tilde{\mathbf{X}}$, $\mathbf{Y}$] and Cov[$\tilde{\mathbf{Y}}$, $\mathbf{X}$] are not zero, as it was assumed for deriving equation (A2), then, in addition to the terms already written in the right-hand side of (A2), we will now have the following nonzero terms:

$$\mathrm{Cov}[\tilde{\mathbf{X}}, \mathbf{h}\,\mathbf{Y}] + \mathrm{Cov}[\mathbf{h}\,\mathbf{Y}, \tilde{\mathbf{X}}] - \mathrm{Cov}[\mathbf{h}\,\tilde{\mathbf{Y}}, \mathbf{h}\,\mathbf{Y}] - \mathrm{Cov}[\mathbf{h}\,\mathbf{Y}, \mathbf{h}\,\tilde{\mathbf{Y}}] \tag{A23}$$

or, equivalently,

$$\mathrm{Cov}[\tilde{\mathbf{X}}, \mathbf{Y}]\,\mathbf{h}^T + \mathbf{h}\,\mathrm{Cov}[\mathbf{Y}, \tilde{\mathbf{X}}] - \mathbf{h}\,\mathrm{Cov}[\tilde{\mathbf{Y}}, \mathbf{Y}]\,\mathbf{h}^T - \mathbf{h}\,\mathrm{Cov}[\mathbf{Y}, \tilde{\mathbf{Y}}]\,\mathbf{h}^T \tag{A24}$$

Apparently, under condition (46), the sum of all terms becomes zero.

Similarly, in addition to the terms already written in the right-hand side of (A6), we will now have the following nonzero terms:

$$\mathrm{Cov}[\tilde{\mathbf{X}}, \mathbf{Y}] - \mathbf{h}\,\mathrm{Cov}[\tilde{\mathbf{Y}}, \mathbf{Y}] \tag{A25}$$

which apparently, under condition (46), add up to zero. Thus, item (a) of the Proposition remains valid.

For the proof of item (b) of the Proposition we have not used any assumption about the dependence of $\mathbf{Y}$ on $\widetilde{\mathbf{X}}$ or $\widetilde{\mathbf{Y}}$; therefore the proof of Appendix A1 remains valid.

Similarly, the second part of the proof of item (c) of the Proposition (that concerning the minimization of the variance of estimation error U) has been derived without any assumption about the dependence of $\mathbf{Y}$ on $\widetilde{\mathbf{X}}$ or $\widetilde{\mathbf{Y}}$ and therefore it remains valid. Because $\text{Var}[X_i \mid \mathbf{Y} = \mathbf{y}]$ cannot be lower than the minimum $\text{Var}[U]$, the introduction of nonnegative correlation of $\mathbf{Y}$ to $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$ cannot improve (i.e., reduce) this conditional variance and thus $\text{Var}[X_i \mid \mathbf{Y} = \mathbf{y}]$ keeps the value given by (8).

A3 Proofs of equations of parameter evaluation (section 4)

A3.1 Equation (31)

Subtracting means from both sides of (3) and then postmultiplying by $(\mathbf{X}_r - E[\mathbf{X}_r])^T$, with $r < s$, we get

$$\text{Cov}[\mathbf{X}_s, \mathbf{X}_r] = \mathbf{a}_s \,\text{Cov}[\mathbf{X}_{s-1}, \mathbf{X}_r] + \mathbf{b}_s \,\text{Cov}[\mathbf{V}_s, \mathbf{X}_r], \quad s > r \tag{A26}$$

Since $s > r$, $\mathbf{V}_s$ and $\mathbf{X}_r$ are uncorrelated and thus the last term of the right-hand side of (A26) is zero. Hence,

$$\boldsymbol{\sigma}_{sr} = \mathbf{a}_s \,\boldsymbol{\sigma}_{s-1,r}, \quad s > r \tag{A27}$$

Applying (A27) recursively (i.e. substituting $\boldsymbol{\sigma}_{s-1,r}$ with $\mathbf{a}_{s-1}\,\boldsymbol{\sigma}_{s-2,r}$, etc.) we get (31).

A3.2 Equation (33)

Subtracting means from both sides of (32) and then postmultiplying by $(\mathbf{X}_r - E[\mathbf{X}_r])^T$, with $r < s$ we get

$$\text{Cov}[\mathbf{X}_s, \mathbf{X}_r] = \mathbf{a}_s \,\text{Cov}[\mathbf{X}_{s-1}, \mathbf{X}_r] + \mathbf{e}_s \,\text{Cov}[\mathbf{X}_{s-2}, \mathbf{X}_r] + \mathbf{b}_s \,\text{Cov}[\mathbf{V}_s, \mathbf{X}_r], \quad s > r \tag{A28}$$

Since $s > r$, $\mathbf{V}_s$ and $\mathbf{X}_r$ are uncorrelated and thus the last term of the right-hand side of (A28) is zero. Hence, (33) is derived.

A3.3 Equation (35)

Writing (34) for $\mathbf{X}_r$, subtracting means from both its sides and then postmultiplying by $(\mathbf{V}_s - E[\mathbf{V}_s])^T$ we get

$$\text{Cov}[\mathbf{X}_r, \mathbf{V}_s] = \mathbf{a}_r \,\text{Cov}[\mathbf{X}_{r-1}, \mathbf{V}_s] + \mathbf{b}_r \,\text{Cov}[\mathbf{V}_r, \mathbf{V}_s] + \mathbf{e}_r \,\text{Cov}[\mathbf{V}_{r-1}, \mathbf{V}_s] \tag{A29}$$

If $s > r$, then $\mathbf{V}_r$ will be uncorrelated to both $\mathbf{X}_s$ and $\mathbf{X}_{s-1}$, and thus all terms of (A29) are zero. If $s = r$, then (A29) yields $\text{Cov}[\mathbf{X}_r, \mathbf{V}_r] = \mathbf{b}_r$, since $\text{Cov}[\mathbf{X}_{r-1}, \mathbf{V}_r] = \text{Cov}[\mathbf{V}_{r-1}, \mathbf{V}_r] = \mathbf{O}$ and $\text{Cov}[\mathbf{V}_r, \mathbf{V}_r] = \mathbf{I}$. Thus,

$$\begin{aligned} &\text{Cov}[\mathbf{X}_r, \mathbf{V}_s] = \mathbf{b}_r, \quad s = r \\ &\text{Cov}[\mathbf{X}_r, \mathbf{V}_s] = \mathbf{O}, \quad s > r \end{aligned} \tag{A30}$$

Subtracting means from both sides of (34) and then postmultiplying by $(\mathbf{X}_r - E[\mathbf{X}_r])^T$, we get

$$\mathrm{Cov}[\mathbf{X}_s, \mathbf{X}_r] = \mathbf{a}_s\, \mathrm{Cov}[\mathbf{X}_{s-1}, \mathbf{X}_r] + \mathbf{b}_s\, \mathrm{Cov}[\mathbf{V}_s, \mathbf{X}_r] + \mathbf{e}_s\, \mathrm{Cov}[\mathbf{V}_{s-1}, \mathbf{X}_r] \qquad \text{(A31)}$$

If $s > r + 1$, then from (A30) both the middle and the last term of the right-hand side of (A31) will be zero and this proves the second of (35). If $s = r + 1$, then from (A30) the middle term of (A31) will be zero but the last term will be $\mathbf{e}_s\, \mathbf{b}_r^T$; this proves the first of (35).

A3.4 Equation (37)

We consider any two variables X_s^l and X_r^j, and their logarithmic transformations $X_s^{l*} := \ln(X_s^l - c_s^l)$ and $X_r^{j*} := \ln(X_r^j - c_r^j)$. Since X_s^{l*} and X_r^{j*} are jointly normal (X_s^l and X_r^j are jointly lognormal), the variable $W^* := X_s^{l*} + X_r^{j*}$ has normal distribution with $\eta := E[W^*] = \mu_s^{l*} + \mu_r^{j*}$ and $\theta := \mathrm{Var}[W^*] = \sigma_{ss}^{ll*} + \sigma_{rr}^{jj*} + 2\,\sigma_{sr}^{lj*}$. Taking the characteristic function $\Phi(\omega)$ of the normal distribution (the Fourier transform of its density), i.e.,

$$\Phi(\omega) := E[\exp(\mathrm{i}\,\omega\, W^*)] = \exp[\mathrm{i}\,\eta\,\omega - \theta\,\omega^2 / 2] \qquad \text{(A32)}$$

where $\mathrm{i} = \sqrt{-1}$, and setting $\omega = -\mathrm{i}$ we get

$$E[\exp(W^*)] = \exp[\eta + \theta / 2] = \exp[\mu_s^{l*} + \mu_r^{j*} + (\sigma_{ss}^{ll*} + \sigma_{rr}^{jj*}) / 2 + \sigma_{sr}^{lj*}] \qquad \text{(A33)}$$

Combining (38) we get

$$E[\exp(W^*)] = (\mu_s^l - c_s^l)\,(\mu_r^j - c_r^j)\,\exp(\sigma_{sr}^{lj*}) \qquad \text{(A34)}$$

Besides,

$$E[\exp(W^*)] = E[\exp(X_s^{l*})\,\exp(X_r^{j*})] = E[(X_s^l - c_s^l)\,(X_r^j - c_r^j)] \qquad \text{(A35)}$$

and

$$\mathrm{Cov}[X_s^l, X_r^j] = \mathrm{Cov}[X_s^l - c_s^l, X_r^j - c_r^j] = E[(X_s^l - c_s^l)\,(X_r^j - c_r^j)] - E[X_s^l - c_s^l]\,\mathrm{E}[X_r^j - c_r^j] \qquad \text{(A36)}$$

Thus,

$$\mathrm{Cov}[X_s^l, X_r^j] = (\mu_s^l - c_s^l)\,(\mu_r^j - c_r^j)\,\exp(\sigma_{sr}^{lj*}) - (\mu_s^l - c_s^l)\,(\mu_r^j - c_r^j) \qquad \text{(A37)}$$

which proves (37).